



PALSfit

A computer program for analysing positron lifetime spectra

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PALSfit: A computer program for analysing positron lifetime spectra

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Risø-R-1652(EN)

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lifetime spectra

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Abstract:

A Windows based computer program PALS*fit* has been developed for analysing positron lifetime spectra. The program is built upon the well tested PATFIT package. The present document describes the mathematical foundation of the PALS*fit* model as well as a number of features of the program. The two cornerstones in PALS*fit* are the following least-squares fitting modules: *PositronFit*, which extracts lifetimes and intensities from the measured lifetime spectra, and *ResolutionFit*, which determines the time resolution function from lifetime spectra that have been recorded for this purpose. The PALS*fit* software is available from Risø DTU (www.palsfit.dk). At this web-site there are also links to test examples and hands-on description of the windows interface.

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Contents

1	Introduction and overview	5
2	About PALSfit	8
2.1	General fitting criterion	8
2.2	PALSfit input	8
2.3	PALSfit output in general	9
2.4	The <i>PositronFit</i> model	9
2.5	The Main Output from <i>PositronFit</i>	11
2.6	The <i>ResolutionFit</i> model	13
2.7	The Main Output from <i>ResolutionFit</i>	13
3	Experience with PALSfit	15
3.1	<i>PositronFit</i> experience	15
3.2	<i>ResolutionFit</i> experience	16
4	Least-squares fit and statistics	18
4.1	Unconstrained NLLS data fitting	18
4.2	Constraints	20
4.3	Statistical analysis	21
4.4	Statistical interpretations	23
4.5	Marquardt minimization	25
4.6	Separable least-squares technique	27
4.7	Various mathematical and numerical issues	28
5	Mathematical model details	33
5.1	<i>PositronFit</i>	33
5.2	<i>ResolutionFit</i>	35
6	Control files	35
6.1	<i>PositronFit</i> control file	36
6.2	<i>ResolutionFit</i> control file	39
	References	40

1 Introduction and overview

An important aspect of doing experiments by positron annihilation lifetime spectroscopy (PALS) is carrying out an analysis of the measured spectra to extract physically meaningful parameters. A number of computer programs have been developed over the last many years by various authors for this purpose. At our laboratory we have concentrated on developing programs for least squares fitting of such spectra.

PALS*fit* is our newest program of this kind. It is based on the well tested PATFIT software [1, 2], which has been used extensively by the positron annihilation community. A brief, preliminary description of the PALS*fit* program was given in [3]. Taking advantage of a new Graphical User Interface (GUI) for Windows, a number of user friendly facilities have been incorporated, in particular graphics displays.

The two cornerstones in PALS*fit* are the following least-squares fitting modules:

- *PositronFit* extracts lifetimes and intensities from lifetime spectra.
- *ResolutionFit* determines the lifetime spectrometer time resolution function to be used in *PositronFit* analyses.

Correspondingly PALS*fit* may run in either of two modes, producing a *PositronFit* analysis or a *ResolutionFit* analysis, respectively.

Common for both modules is that a model function will be fitted to a measured spectrum. The model function consists of a sum of decaying exponentials convoluted with a time resolution function, plus a constant background. The time resolution function is described by a sum of Gaussians which may be displaced with respect to each other. Various types of constraints may be imposed on the fitting parameters.

The structure of the PALS*fit* program is sketched in Fig. 1 p. 6 and an example of a window in the actual GUI of PALS*fit* is shown in Fig. 2 on p. 7.

In *ResolutionFit*, parameters determining the shape of the resolution function can be fitted, normally by analysing lifetime spectra which contain mainly one component. The extracted resolution curve may then be used in *PositronFit* to analyse more complicated spectra. PALS*fit* provides facilities for accomplishing this process.

Note that in *PositronFit* the shape of the resolution function is fixed. A correction for positrons annihilating outside the sample can be made during the *PositronFit* analysis [1].

In Chapter 2 we give a presentation of *PositronFit* and *ResolutionFit*. This includes a short overview of the mathematical models as well as output examples. In Chapter 3 we convey some experiences we and others have gained with PALS*fit* and its predecessors. Chapters 4 and 5 contain mathematical and statistical details, while Chapter 6 displays examples of so-called control files produced by PALS*fit* (either the *PositronFit* or *ResolutionFit* module).

PALS*fit* is available from Risø DTU (by downloading from www.palsfit.dk).

A contemporary edition of the PATFIT package, roughly equivalent to PALS*fit* without its GUI, is available too. It contains command-driven versions of *PositronFit* and *ResolutionFit* and might be useful for batch processing or in a Linux environment.

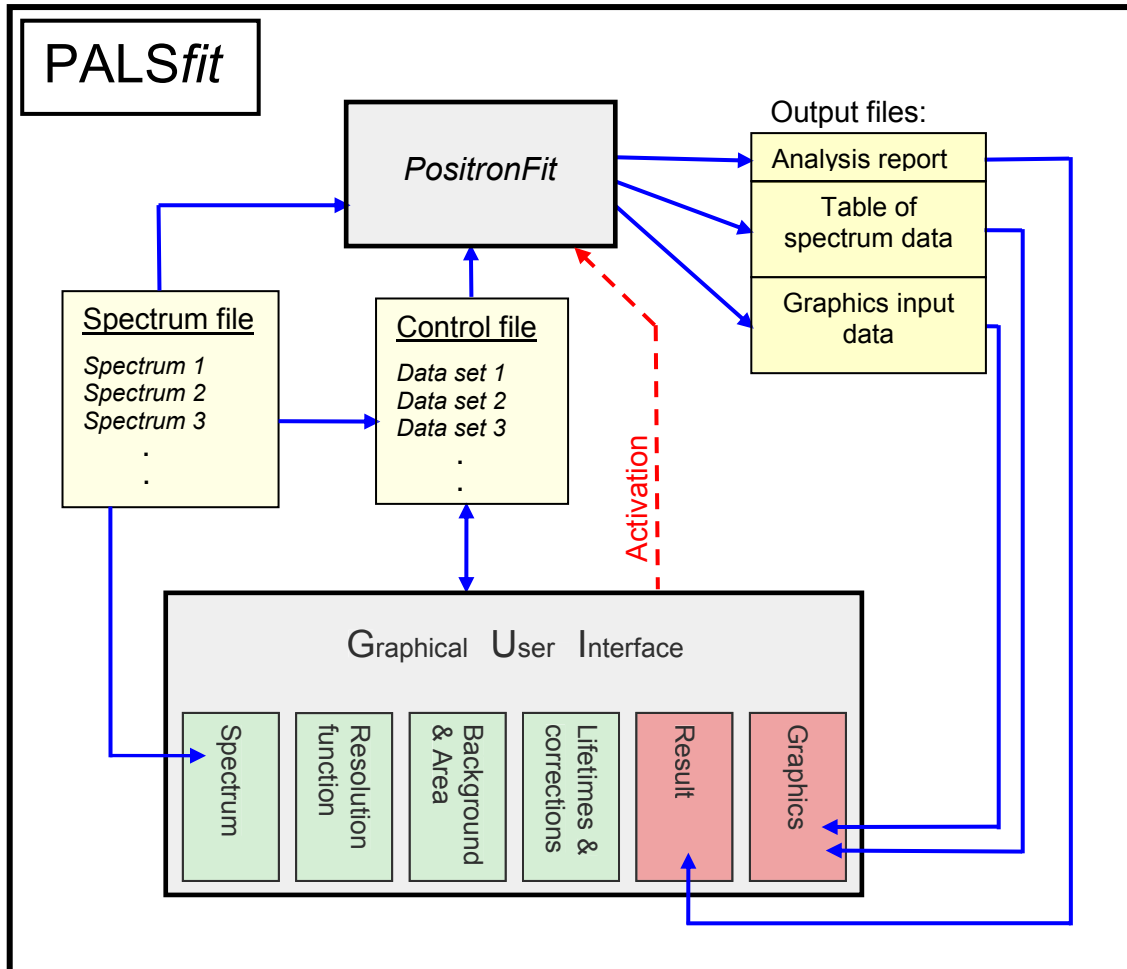


Fig. 1. Schematic illustration of the main structure of the PALSfit program, using the *PositronFit* module as example. PALSfit consists of a Graphical User Interface (GUI) and the PATFIT fitting programs (*PositronFit* or *ResolutionFit*). The program components communicate via a number of files (yellow boxes). The GUI consists of input sections (green) and output sections (red). Based on the data that are entered in the four (green) input sections, the GUI also creates a Control file (consisting of one or more data sets) that provides the input parameters for the *PositronFit* fitting program. *PositronFit* generates three different files, which are used by two (red) output sections of the GUI to display the result of the analysis in numerical or graphical form. Communication between programs is indicated by arrows: blue solid-line arrows show data flow, red dashed-line arrow shows control of *PositronFit* by the GUI. An example of the appearance of an actual GUI is shown in Fig. 2.

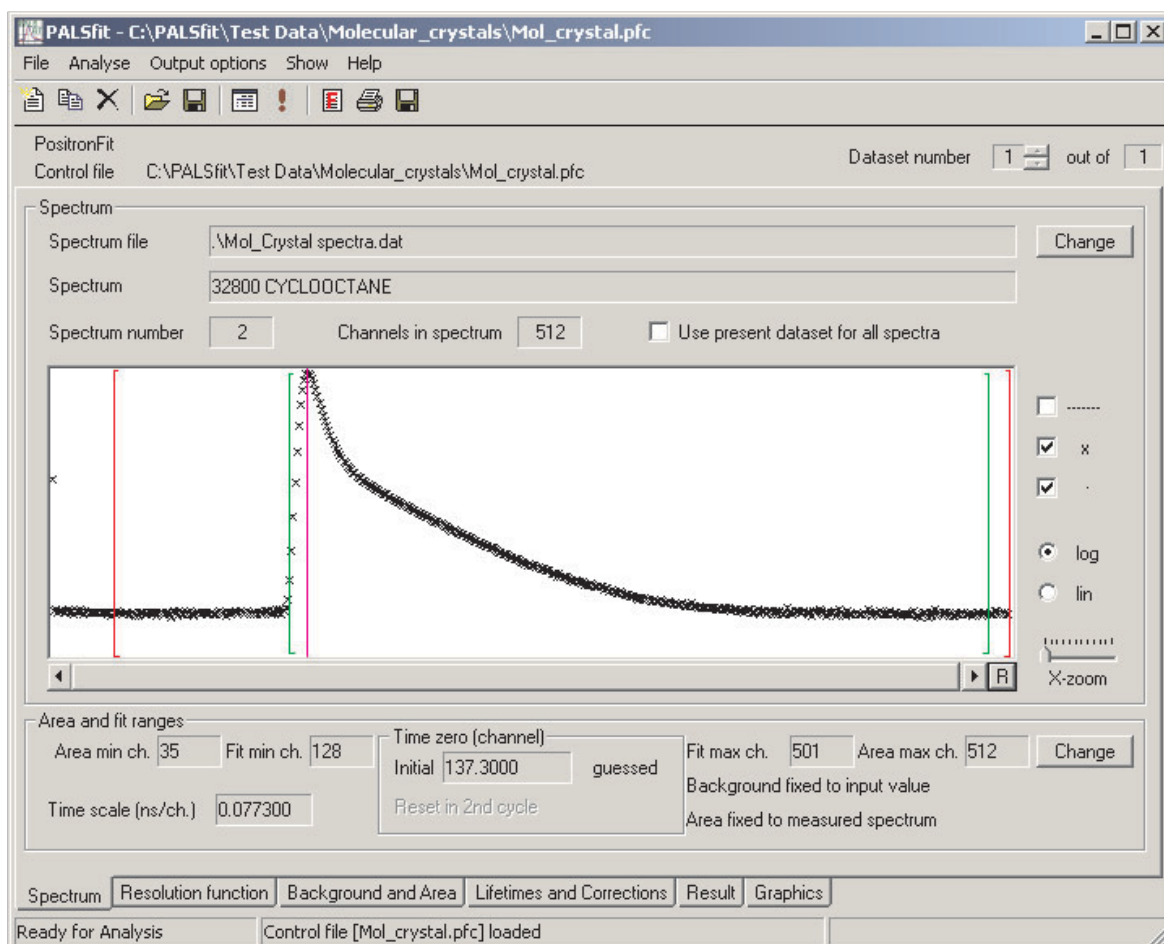


Fig. 2. An example of a window in the actual Graphical User Interface (GUI) of the PALSfit program, which shows a spectrum to be analysed, some of the input parameters for the analysis as well as icons, buttons, menus and tabs that are used to define the analysis and to display the results in numerical or graphical form. The six tabs, which open four windows for data input and two windows for display of results, are the ones which are indicated in the GUI in Fig. 1.

2 About PALSfit

2.1 General fitting criterion

Common for *PositronFit* and *ResolutionFit* is that they fit a parameterized model function to a distribution (a “spectrum”) of experimental data values y_i . In the actual case these are count numbers which are recorded in “channels” and are obeying Poisson statistics. We use the least-squares criterion, i.e. we seek values of the k model parameters b_1, \dots, b_k that minimizes

$$\phi \equiv \sum_{i=1}^n w_i (y_i - f_i(b_1, \dots, b_k))^2 \quad (1)$$

where n is the number of data values, $f_i(b_1, \dots, b_k)$ the model prediction corresponding to data value no. i , and w_i a fixed weight attached to i ; in this work we use “statistical weighting”,

$$w_i = \frac{1}{s_i^2} \quad (2)$$

where s_i^2 is the estimated variance of y_i ($= y_i$ itself or the model function f_i for unnormalized counts).

As some of the parameters enter our models nonlinearly, we must use an iterative fitting technique. In PALSfit we use separable least-square methods to obtain the parameter estimates. Details of the solution methods and the statistical inferences are given in Chapter 4. As a result of the calculations, a number of fitting parameters are estimated that characterize the fitted model function and hence the measured spectrum (e.g. lifetimes and intensities). A number of different constraints may be imposed on the fitting parameters. The two most important types of constraints are that 1) a parameter can be fixed to a certain value, and 2) a linear combination of intensities is put equal to zero (this latter constraint can be used to fix the ratio of intensities).

2.2 PALSfit input

PALSfit requires — together with the spectrum to be analysed — a set of input data, e.g. some characteristic parameters of the lifetime spectrometer, guesses of the parameters to be fitted, and possible constraints on these parameters. These data are organised in block structured *control data sets* which in turn are collected in *control files*. A control file may thus contain several control data sets.

PALSfit can interactively generate and/or edit the control file for either *PositronFit* or *ResolutionFit*. Previously generated control data can be used as default input values. A number of checks on the consistency of the generated control data are built into PALSfit.

Since PALSfit is largely self-explanatory regarding input editing, we shall not discuss the input facilities here but refer to the hands-on description of PALSfit, which can be accessed by a link in the website www.palsfit.dk.

We recommend that the user reads the model descriptions in the following sections and consults Chapter 6, where the structure of the *PositronFit* and *ResolutionFit* control files is discussed. This will give a clear impression of the input possibilities in PALSfit.

Note that control files may also be produced and edited by other means than PALSfit.

2.3 PALSfit output in general

After a successful *PositronFit* or *ResolutionFit* analysis PALSfit produces three files: an analysis report (result file), a graphics file and a table file. The analysis report (result file) has the following contents:

- a) An edited result section, which is the *Main Output* for the analysis. It contains the final estimates of the fitting parameters and their standard deviations. In addition, all the guessed input parameters as well as information on constraints are quoted. Furthermore, three statistical numbers, “chi-square”, “reduced chi-square”, and “significance of imperfect model” are shown. They inform about the agreement between the measured spectrum and the model function (Section 4.4). A few key numbers are displayed for quick reference, giving the number of components and the various types of constraints; they are identified by letters or abbreviations.
- b) An input echo (optional). This is a raw copy of all the input data contained in the *Control data set*.
- c) Fitting parameters after each iteration (optional). The parameters shown are internal; after convergence they may need a transformation prior to presentation in the Main Output.
- d) An estimated correlation matrix for the parameters (optional). This matrix and its interpretation is discussed in Section 4.4.

As indicated above, the outputs b)–d) are optional, while the Main Output is always produced.

The graphics file contains data necessary for the generation of plots of measured and fitted spectra and the table file contains a table with the measured and the fitted spectrum values as a function of the channel number.

Apart from showing a result file, PALSfit displays data and results in graphical form on the screen. The PALSfit plots include measured or fitted spectra or superpositions of both, as well as deviations between them.

2.4 The *PositronFit* model

The *PositronFit* (and *ResolutionFit*) model function consists of a sum of decaying exponentials convoluted with the resolution function of the lifetime spectrometer, plus a constant background. Let t be the time, k_0 the number of lifetime components, a_j the decay function for component j , R the time-resolution function, and B the background. The resulting expression is given in full detail in Section 5.1; here we state the model in an annotated form using the symbol $*$ for convolution:

$$f(t) = \sum_{j=1}^{k_0} (a_j * R)(t) + B \quad (3)$$

where

$$a_j(t) = \begin{cases} A_j \exp(-t/\tau_j), & t > 0 \\ 0, & t < 0 \end{cases} \quad (4)$$

In (4) τ_j is the mean lifetime of the j th component, and A_j a pre-exponential factor ($A_j\tau_j$ is the “area” of the component.) We assume, furthermore, that R is given by a sum of k_g Gaussians which may be displaced with respect to each other:

$$R(t) = \sum_{p=1}^{k_g} \omega_p G_p(t) \quad (5)$$

where

$$G_p(t) = \frac{1}{\sqrt{2\pi}\sigma_p} \exp\left(-\frac{(t - T_0 - \Delta t_p)^2}{2\sigma_p^2}\right) \quad (6)$$

and

$$\sum_{p=1}^{k_g} \omega_p = 1 \quad (7)$$

The Gaussian (6) is centered around channel number $T_0 + \Delta t_p$, where T_0 corresponds to a reference channel number (called “time-zero”) and Δt_p is a displacement. The standard deviation σ_p of the Gaussian is related to its Full Width at Half Maximum by

$$\text{FWHM}_p = 2\sqrt{2 \ln 2} \sigma_p \quad (8)$$

The curve given by (3) is a continuous curve, while the spectra normally are recorded in channels of a multichannel analyser. For proper comparison, the curve (3) shall therefore be transformed into a histogram by integration over intervals each being one channel wide. The model count distribution which is actually fitted to the measured spectrum is therefore given by:

$$f_i = \int_{t_i}^{t_{i+1}} f(t) dt \quad (9)$$

where t_i is the value of t at the common boundary of channel Nos. $i - 1$ and i . Here we assume equidistant channel widths and measure the time in units of channels such that $t_{i+1} - t_i = 1$; in this way the integrated count (9) becomes a channel average.

As the result, we obtain a model for the least-squares analysis of the form

$$f_i = \sum_{j=1}^{k_0} F_{ij} + B \quad (10)$$

where F_{ij} is the contribution from lifetime component j in spectral channel i . (We relegate the full write-up of F_{ij} to Section 5.1.) Note that f_i in (9) and (10) corresponds to $f_i(b_1, \dots, b_k)$ in Section 2.1, formula (1).

The fitting parameters in *PositronFit* are the lifetimes (τ_j), the relative intensities defined as

$$I_j = \frac{A_j \tau_j}{\sum_{k=1}^{k_0} A_k \tau_k} \quad (11)$$

the time-zero (T_0), and the background (B). Each of these parameters may be fixed to a chosen value. In another type of constraint you may put one or more linear combinations of intensities equal to zero in the fitting, i.e.

$$\sum_{j=1}^{k_0} h_{lj} I_j = 0 \quad (12)$$

These constraints can be used to fix ratios of intensities. Finally, it is possible to fix the total area of the spectrum in the fitting,

$$\sum_{j=1}^{k_0} A_j \tau_j + \text{background area} = \text{constant} \quad (13)$$

This may be a useful option if, for example, the peak region of the measured spectrum is not included in the analysis.

Normally in an experiment a fraction α of the positrons will not annihilate in the sample, but for example in the source or at surfaces. In *PositronFit* it is possible to make a correction for this (“source correction”). First, the raw data are fitted in a first iteration cycle. Then, the spectrum of the source correction is subtracted and the corrected spectrum fitted again in a second iteration cycle. In this second cycle it is optional to choose another number

of lifetime components as well as type and number of constraints than are used for the first iteration cycle. The source correction spectrum f_i^s itself is composed of k_s lifetime components and expressed in analogy with (10) (with $B = 0$) as follows:

$$f_i^s = \sum_{j=1}^{k_s} F_{ij}^s \quad (14)$$

If τ_j^s and A_j^s are the lifetime and pre-exponential factor, respectively, of source-correction component j , then

$$\sum_{j=1}^{k_s} A_j^s \tau_j^s = \alpha \sum_{j=1}^{k_0} A_j \tau_j \quad (15)$$

Between the two iteration cycles we take the opportunity to improve the estimates of s_i^2 in (2) by using model-predicted counts instead of y_i . This weight-smoothing procedure is invoked even when there is no source correction, such that there always are two iteration cycles.

The necessary mathematical processing of the *PositronFit* model for the least-squares analysis is outlined in Section 5.1.

2.5 The Main Output from *PositronFit*

In the following we give an example of the Main Output part of a *PositronFit* analysis report produced by *PALSfit*, with a brief explanation of its contents (for more details about the input possibilities consult Section 6.1):

```
PALSfit - Version 1.64 27-jan-2009 - Licensed to Jens V. Olsen
Input file: C:\PALSfit-test\Molecular crystal\Mol_crystal.pfc

P O S I T R O N F I T . Version 1.64      Job time 14:28:21.52 27-JAN-09
*****
32800 CYCLOOCTANE
*****
Data set 1                                L T I B Z A G
                                           3 1 1 2 0 1 3

Time scale  ns/channel      : 0.077300
Area range  starts in ch.   35 and ends in ch. 512
Fit range   starts in ch.  136 and ends in ch. 501

Resolution  FWHM (ns)      : 0.5395    0.3539    0.4036
Function     Intensities (%) : 20.0000   20.0000   60.0000
              Shifts (ns)   : 0.0412    -0.0749    0.0000

----- I n i t i a l   P a r a m e t e r s -----
              Time-zero (ch.no): 137.3000G
              Lifetimes (ns)   : 0.2300F   0.4000G   2.6500G
              Intensities (%)  :                      28.0000F

Background  Fixed input value: 680.0000
Area        Fixed to measured spectrum ch. 35 -> 512      : 9.05171E+06

----- R e s u l t s   b e f o r e   S o u r c e   C o r r e c t i o n -----
Convergence obtained after 6 iterations
Chi-square = 549.77 with 362 degrees of freedom
              Lifetimes (ns)   : 0.2300F   0.4401   2.6696
              Intensities (%)  : 30.6011   41.3989   28.0000F
Time-zero    Channel number   : 136.2960
Total-area   From fit         : 9.05171E+06      From table : 9.05171E+06

----- S o u r c e   C o r r e c t i o n -----
              Lifetimes (ns)   : 0.3840    0.9056
              Intensities (%)  : 8.0000    0.4527
              Total (%)       : 100.0000
```

```

----- 2 n d c y c l e   P a r a m e t e r s -----
      Lifetimes (ns)   :    0.1200G   0.3600G   1.2000G   2.8000G
      Lin.comb.coeff.  :   -3.0000    0.0000    1.0000    1.0000

Area      Fixed to measured spectrum ch. 35 -> 512      : 8.31407E+06

##### F i n a l   R e s u l t s #####
Data set 1                                     L T I B Z A G
                                           4 0 -1 2 0 1 3

Convergence obtained after 11 iterations
Chi-square = 439.01 with 359 degrees of freedom
Reduced chi-square = Chi-square/dof = 1.223 with std deviation 0.075
Significance of imperfect model = 99.76 %

      Lifetimes (ns)   :    0.1373    0.3564    1.2100    2.7600
      Std deviations   :    0.0048    0.0011    0.0972    0.0154

      Intensities(%) LC:   11.1027   55.5892    5.0721   28.2361
      Std deviations   :    0.0959    0.3834    0.2999    0.4630

Background  Counts/channel : 680.0000
            Std deviations  :      fixed

Time-zero   Channel number : 136.3540
            Std deviations  :    0.0113

Total-area  From fit       : 8.31407E+06      From table : 8.31407E+06

##### P o s i t r o n F i t #####

```

This output was obtained by running *PALSfit* with the control data set listed in Section 6.1. It does not represent a typical analysis of a spectrum, but is rather meant to illustrate a number of features of the program.

After a heading which contains the spectrum headline the key numbers are displayed in the upper right hand corner. L indicates the number of lifetime components (k_0), T the number of fixed lifetimes, I the number and type of intensity constraints (a positive number for fixed intensities, a negative number for linear combinations of intensities, i.e. the number m , Section 6.1, Block 5), B the type of background constraint (KB, Section 6.1, Block 6), Z whether time-zero is free or fixed (0 = free, 1 = fixed), A the type of area constraint (KAR, Section 6.1, Block 7), and G the number of Gaussians used to describe the time resolution function (k_g). The rest of the upper part of the output reproduces various input parameters, in particular those for the resolution function, the shape of which is fixed, and the initial values (G for guessed and F for fixed) of the fitting parameters.

The next part (results before source correction) contains the outcome of the first iteration cycle. If convergence could not be obtained, a message will be given and the iteration procedure discontinued, but still the obtained results are presented. Then follows information about the goodness of the fit (Section 4.4).

The next part (source correction) shows the parameters of the chosen source correction, which accounts for those positrons that annihilate outside the sample, as well as optional initial values of the fitting parameters for the second iteration cycle.

The “Final Results” part contains the number of iterations in the final cycle, followed by three lines with information about the goodness of the fit (Section 4.4). Then follows a survey of the final estimates of the fitted (and fixed) parameters and their standard deviations. The “LC” in the intensity line indicates that we have intensity constraints of the linear-combination type (cf. the negative I in the upper right hand corner). The “total area from fit” is calculated as $\sum_j A_j \tau_j$ plus the background inside the “area range” specified in the beginning of the Main Output. The “total area from table” is the total number of counts in the (source corrected) measured spectrum inside the “area range”.

2.6 The *ResolutionFit* model

The *ResolutionFit* model function is the same as for *PositronFit*, Eqs. (3–10). A few additional formulas relevant to *ResolutionFit* are given in Section 5.2. The purpose of *ResolutionFit* is to extract the shape of the resolution function. The widths and displacements (Eqs. (8) and (6)) of the Gaussians in the resolution function are therefore included as fitting parameters. However, in order not to have too many fitting parameters (which may lead to ill-defined estimates of the parameters) the intensities of the Gaussians are fixed parameters. For the same reason it is normally advisable to determine resolution functions by fitting only simple lifetime spectra, i.e. spectra containing only one major lifetime component. The extracted resolution function may then be used in *PositronFit* to analyse more complicated spectra. Along the same line, *ResolutionFit* does not include as many features as does *PositronFit*, e.g. there is no source correction and there are no constraints possible on time-zero or on the area. However, the background can be free or fixed, just like in *PositronFit*.

Hence, the fitting parameters in *ResolutionFit* are the lifetimes (τ_j), their relative intensities (I_j), the background (B), the time-zero (T_0), and the widths and displacements of the Gaussians in the resolution function. Each of these parameters, except T_0 , may be constrained to a fixed value and, as in *PositronFit*, linear combinations of lifetime intensities may be constrained to zero in the fitting.

The various input options will be illustrated in the sample output in the next section.

Like *PositronFit*, *ResolutionFit* uses weight smoothing followed by a second iteration cycle.

2.7 The Main Output from *ResolutionFit*

In the following we give an example of the Main Output part of a *ResolutionFit* analysis report produced by *PALSfit*, with a brief explanation of its contents (for more details about the input possibilities consult Section 6.2):

```
PALSfit - Version 1.64 27-jan-2009 - Licensed to Jens V. Olsen
Input file: C:\PALSfit-test\Metal defects\Cu-ref.rfc

R E S O L U T I O N F I T Version 1.64    Job time 14:29:13.48 27-JAN-09
*****
39699 CU-ANNEALED
*****
Data set 1
Time scale   ns/channel       :   0.013400
Area range   starts in ch.     5 and ends in ch. 1000
Fit range     starts in ch.    140 and ends in ch. 500

Initial      FWHM (ns)        :   0.2600G   0.3000G   0.4000G
Resolution   Intensities (%)  :   77.0000   19.0000   4.0000
Function      Shifts (ns)     :   0.0000F   0.0223G  -0.0462G

Other init.   Time-zero (ch.no): 173.0000
Parameters    Lifetimes (ns)  :   0.1100F   0.1800F   0.4000G
Background    fixed to mean from ch. 650 to ch. 1000 = 12.0342

##### F i n a l   R e s u l t s #####

Convergence obtained after 28 iterations
Chi-square = 315.57 with 351 degrees of freedom
Reduced chi-square = Chi-square/dof = 0.899 with std deviation 0.075
Significance of imperfect model = 8.69 %
-----
Resolution function:                                     G W S
                                                         3 0 1
                FWHM (ns)      :   0.2396   0.3099   0.2845
```

```

Std deviations      :      0.0007      0.0143      0.0104

Intensities (%)    :      77.0000     19.0000      4.0000

Shifts (ns)        :      0.0000      0.0574     -0.1900
Std deviations      :      fixed      0.0097      0.0056
-----
Lifetime components:
L T I
3 2 0

Lifetimes (ns)     :      0.1100      0.1800      0.4030
Std deviations      :      fixed      fixed      0.0143

Intensities (%)    :      86.4617     10.7560      2.7823
Std deviations      :      1.0779      1.3443      0.2911
-----
Background:
B
1

Counts/channel     :      12.0342
Std deviation       :      mean
-----
Time-zero   Channel number :      171.3137
Std deviations :      0.1383

Total-area   From fit      : 1.93637E+06      From table : 1.93653E+06

Shape parameters for resolution curve (nsec):
N      2      5      10      30      100      300      1000
FW at  1/N  0.2552  0.3977  0.4868  0.6216  0.7659  0.8825  0.9926
MIDP at 1/N  0.0006  0.0010  0.0004 -0.0043 -0.0135 -0.0195 -0.0228

Peak position of resolution curve: Channel # 171.6090
##### Resolution Fit #####

```

This output was obtained by running *PALSfit* with the control data set listed in Section 6.2.

After a heading which includes the spectrum headline, the upper part of the output reproduces various input parameters in a way that is very similar to the *PositronFit* output. The important difference is that in *ResolutionFit* all the FWHMs and all of the displacements (called “shifts” for brevity) except one, may be fitting parameters. In addition, the background is displayed as well as the channels between which it is calculated, if the background is fixed to the mean value between these channel limits.

In the “Final Results” part the number of iterations used to obtain convergence is given first. The next three lines contain information about how good the fit is (for definition of the terms see Section 4.4).

The main part of the output, i.e. the estimated values of the fitted (and fixed) parameters and their standard deviations, follows next (for fixed parameters **FIXED** is written instead of the standard deviation). This part is divided into three, one giving the parameters for the resolution curve, one with the lifetimes and their intensities, and one showing the background. Each part has one or three key numbers displayed in the upper right hand corner. For the resolution function the **G** indicates the number of Gaussians (k_g), **W** the number of fixed widths, and **S** the number of fixed displacements (shifts). For the lifetime components the **L** indicates the number of these (k_0), **T** the number of fixed lifetimes, and **I** the number and type of intensity constraints. As in *PositronFit*, a positive value means fixed intensities, while a negative value indicates constraints on linear combinations of intensities, the absolute value giving the number of constraints. Next follows the background output, where **B** indicates the type of background constraint (KB, Section 6.2, Block 6), and after the estimated time-zero the “total area from fit” and “total area from table” are given, both calculated as in *PositronFit*. Finally, for easy comparison of the extracted resolution curve with other such curves, a table of the full width of this curve at different fractions of its peak value is displayed, as well as of the midpoints of the curve compared to the peak position. The latter number clearly shows possible asymmetries in the resolution curve.

Also the channel number of the position of the peak (maximum value) of the resolution curve is given.

3 Experience with PALS*fit*

In this section we shall give a short account of some of the experiences we (and others) have had with PALS*fit* and its predecessor versions, specifically the program components *PositronFit* and *ResolutionFit*. In general, these fitting programs have proved to be very reliable and easy to use. Further discussion can be found in [1].

The aim of fitting measured spectra will normally be to extract as much information as possible from the spectra. This often entails that one tries to resolve as many lifetime components as possible. However, this has to be done with great care. Because of the correlations between the fitting parameters, and between the fitting parameters and other input parameters, the final estimates of the parameters may be very sensitive to small uncertainties in the input parameters. Therefore, in general, extreme caution should be exercised in the interpretation of the fitted parameters. This is further discussed in e.g. [4, 5, 6, 7, 8, 9, 10, 11]. In this connection, an advantage of the software is the possibility of various types of constraints which makes it possible to select meaningful numbers and types of fitting parameters.

3.1 *PositronFit* experience

The experience gained with *PositronFit* over a number of years shows that in metallic systems with lifetimes in the range 0.1 – 0.5 ns it is possible to obtain information about at most three lifetime components in unconstrained analyses [12, 13, 14] while in some insulators where positronium is formed, up to four components (unconstrained analysis) may be extracted e.g. [15, 16, 17, 18]. (This does not mean of course that the spectra cannot be composed of more components than these numbers. This problem is briefly discussed in, e.g. [4, 10]. Various other aspects of the analysis of positron lifetime spectra are discussed in for example [19, 20, 21, 22]). In this connection it is very useful to be able to change the number of components from the first to the second iteration cycle. In this way, the spectrum can be fitted with two different numbers of components within the same analysis (it is also advantageous to use this feature when a source correction removes, e.g., a long-lived lifetime component from the raw spectrum).

In our experience *PositronFit* always produces the same estimates of the fitted parameters after convergence, irrespective of the initial guesses (except in some extreme cases). However, others have informed us that for spectra containing very many counts (of the order of 10^7) one may obtain different results, depending upon the initial guesses on the fitting parameters, i.e. local minima exist in the χ^2 as function of the fitting parameters; these minima are often quite shallow. When this happens, *PositronFit* as well as most other least-squares fitting codes are in trouble, because they just find some local minimum. From a single fitting you cannot know whether the absolute minimum in the parameter space has been found. The problem of “global minimization” is much harder to solve, but even if we could locate the deepest minimum we would have no guarantee that this would give the “best” parameter values from a physical point of view. In such cases it may be necessary to make several analyses of each spectrum with different initial parameter guesses or measure more than one spectrum under the same conditions, until enough experience has been gained about the analysis behaviour for a certain type of spectra.

3.2 *ResolutionFit* experience

When using a software component as *ResolutionFit* an important question of course is whether it is possible in practice to separate the resolution function reliably from the lifetime components. Our experience and those of others [10, 18, 23, 24, 25] suggest that this separation is possible, although in general great care is necessary to obtain well-defined results [1, 10]. The reason for this is the same as mentioned above, viz. that more than one minimum for χ^2 may exist.

From a practical point of view the question arises as to whether there is too strong correlation between some of the parameters defining the resolution function and the lifetime parameters, in particular when three Gaussians (or more) are used to describe the resolution function. As in the example used in this report (Section 2.7), we have often measured annealed copper in order to deduce the resolution function. Even with different settings of the lifetime spectrometer, the copper lifetime normally comes out from a *ResolutionFit* analysis within a few ps (statistical scatter) of 110 ps (in agreement with results of others, e.g. [26]). Thus, the lifetime is well-defined and separable from the resolution function, even though many parameters are free to vary in the fitting procedure. However, because of the many parameters used to describe the resolution function, one frequently experiences that two (or more) different sets of resolution function parameters may be obtained from the same spectrum in different analyses, if different initial guesses are applied. The lifetimes and intensities come out essentially the same in the different analyses, the fits are almost equally good, and a comparison of the widths at the various heights of the resolution curves obtained in the analyses show that they are essentially identical. Thus, in spite of the many fitting parameters (i.e. so many that the same resolution curve may be described by more than one set of parameters), it still seems possible to separate the lifetimes and resolution function reliably, at least when the lifetime spectrum contains a short-lived component of about 90 % intensity or more.

On the other hand, one cannot be sure that the lifetimes can always be separated from the resolution function easily. If, for example, the initial guesses for the fitting parameters are far from the correct parameters, the result of the fitting may be that, for instance, the fitted resolution function is strongly asymmetrical thereby describing in part the slope of the spectrum which arises from the shorter lifetime component. This latter component will then have a shorter lifetime than the correct one. Such cases — where the resolution function parameters will be strongly correlated to the main lifetime — will be more likely the shorter the lifetime is and the broader the resolution function is.

In principle, it is impossible from the analysis alone to decide whether lifetimes and resolution function are properly separated. However, in practice it will normally be feasible. If the main lifetime and the resolution curve parameters are strongly correlated, it is an indication that they are not properly separated. This correlation may be seen by looking for the changes in the lifetimes or resolution function when a small change is made in one of the resolution function parameters (intensity or one of the fitting parameters using a constraint). Other indications that the lifetimes and resolution function are not properly separated will be that the resulting lifetime deviates appreciably from established values for the particular material or that the half width of the resolution function deviates clearly from the width measured directly with, e.g., a Co-60 source. If the lifetime and the resolution function cannot be separated without large uncertainties on both, one may have to constrain the lifetime to an average or otherwise determined value. Thus, it will always be possible to extract a resolution function from a suitably chosen lifetime spectrum.

A separate question is whether a sum of Gaussians can give a proper representation of the “true” lifetime spectrometer resolution curve, or if some other functional form, e.g., a Gaussian convoluted with two exponentials [23, 25], is better. Of course, it will depend on the detailed shape of the spectrometer resolution curve, but practical experience seems

to show that the two descriptions give only small differences in the extracted shape of the curve [10, 25], and the better the resolution is, the less does a small difference influence the extracted lifetime parameters [10]. The sum-of-Gaussians used in *PALSfit* was chosen because such a sum in principle can represent any shape.

Once a resolution function has been determined from one lifetime measurement, another problem arises: Can this function be used directly for another set of measurements? This problem is not directly related to the software, but we shall discuss it briefly here. The accuracy of the determined resolution function will of course depend on the validity of the basic assumption about the measured lifetime spectrum from which it is extracted. This assumption is that the spectrum consists of a known number of lifetime components (e.g. essentially only one as discussed above) in the form of decaying exponentials convoluted with the resolution function. However, this “ideal” spectrum may be distorted in various ways in a real measurement. For example, instead of one lifetime, the sample may give rise to two almost equal lifetimes which cannot be separated. This will, of course, influence the resulting resolution function. So will source or surface components which cannot be clearly separated from the main component. Another disturbance of the spectrum may be caused by gamma-quanta which are scattered from one detector to the other in the lifetime spectrometer. Such scattered photons may give rise to quite large distortions of a lifetime spectrum. How large they are will depend on energy window settings and source-sample-detector arrangement of the lifetime spectrometer [10, 27, 28]. (Apart from the distortions, these spectrometer characteristics will, of course, also influence the width and shape of the correct resolution function.) In digital lifetime spectrometers that have been developed in recent years it seems possible to discriminate more effectively against some of these undesired distortions of measured spectra [29, 30, 31].

Finally, by means of an example let us briefly outline the way we try to obtain the most accurate resolution function for a set of measurements. Let us say that we do a series of measurements under similar conditions (e.g. an annealing sequence for a defect-containing metal sample). In between we measure an annealed reference sample of the same metal, with — as far as possible — the same source and in the same physical arrangement, and thereby determine the resolution curve. This is done for example on January 2, 7, 12, etc. to keep track of possible small changes due to electronic drift. We then make reasonable interpolations between these resolution curves and use the interpolated values in the analysis of the lifetime spectra for the defect containing samples. Sometimes it is not feasible to always measure the annealed sample in exactly the same physical arrangement as the defect containing sample (for example if the annealing sequence takes place in a cryostat). Then we determine resolution curves from measurements on the annealed sample inside and outside the cryostat (the results may be slightly different) before and after the annealing sequence. The possible time variation (due to electronic drift) of the resolution function is then determined from measurements with the annealed sample outside the cryostat. The same variation is finally applied to the resolution curve valid for measurements inside the cryostat.

As we often use many parameters to describe a resolution function these parameters may appear with rather large scatter. To obtain well-defined variations with time it is often useful in a second analysis of the annealed metal spectra to constrain one or two of the parameters to some average values. With this procedure we believe that we come as close as possible to a reliable resolution function. We are reluctant to determine the resolution function directly from the spectra for the defected metal sample, as we feel that the lack of knowledge of the exact number of lifetime components makes the determination too uncertain.

Let us finally point to one more useful result of an ordinary *ResolutionFit* analysis apart from the extraction of the resolution curve, viz. the determination of the “source correction” (Section 2.4). If the sample gives rise to only one lifetime component, any remaining

components must be due to positrons annihilating outside the sample and is therefore normally considered as a source correction. In the *ResolutionFit* Main Output (Section 2.7) the 0.110 ns is the annealed-Cu lifetime, while the 0.18 ns, 10.756% component is the estimated lifetime and intensity component for the positrons annihilating in the 0.5 mg/cm² nickel foil surrounding the source material. The 0.4030 ns, 2.7823% component, that is determined by the analysis, is believed to arise from positrons annihilating in the NaCl source material and on surfaces. This component may be different for different sources and different samples (due to different backscattering). We consider the latter two components as corrections to the measured spectrum in any subsequent *PositronFit* analysis (when the same source and similar sample material have been used).

4 Least-squares fit and statistics

The first four sections of the present chapter contain general information about nonlinear least-squares (NLLS) methods and their statistical interpretations with relevance for *PALSfit*, but without going into details with the specific models involved; these are discussed in Chapters 2 and 5.

The last three sections are of a more technical nature. Section 4.5 presents essential principles of modern NLLS solution methods. Section 4.6 documents the separable least-squares technique which is of utmost importance for the efficiency and robustness of *PALSfit*, and Section 4.7 contains various mathematical and numerical details.

4.1 Unconstrained NLLS data fitting

We shall first present an overview of the classical unconstrained nonlinear least-squares (NLLS) method for data fitting.

In the classical setup it is assumed that some general model is given,

$$y = f(x; b_1, b_2, \dots, b_k) = f(x; \mathbf{b}) \quad (16)$$

where x and y are the independent and dependent variable, respectively, and $\mathbf{b} = (b_j)$ is a parameter vector with k components. (All vectors in this work are considered as column vectors.) The components b_j may enter linearly or nonlinearly in (16), and so we may talk about linear and nonlinear parameters b_j . Further, a set of n data points (x_i, y_i) ($i = 1, \dots, n$) is given, x_i being the independent and y_i the dependent variable; we shall here introduce the data vector $\mathbf{y} = (y_i)$, also called the *spectrum*. Such a spectrum is usually the result of an experiment. We assume $n \geq k$. According to the least squares principle we should determine $\mathbf{b} \in \mathbb{R}^k$ such that

$$\phi(\mathbf{b}) = \sum_{i=1}^n w_i (y_i - f(x_i; \mathbf{b}))^2 \quad (17)$$

is minimized. The w_i are the *weights* of the data; until further notice they are just arbitrary fixed positive numbers accompanying our sample. (In many applications weights are omitted which corresponds to equal weighting, $w_i = 1$.)

When setting up equation (17) it was assumed that the x_i were sample *points* corresponding to the independent variable x in (16). In practice, however, we do not always have this situation. For example, if x represents time, and the equipment records certain events in fixed time intervals (t_i, t_{i+1}) called *channels*, it would be natural to compare y_i with an average of the model function in (16) over (t_i, t_{i+1}) . Hence it is appropriate to replace (17)

by

$$\phi(\mathbf{b}) = \sum_{i=1}^n w_i (y_i - f_i(\mathbf{b}))^2 \quad (18)$$

In general we shall need a “recipe” $f_i(\mathbf{b})$ to compute the model values to be compared with the data values y_i . No sample points x_i of the independent variable enter (18) directly but may possibly be needed for calculating $f_i(\mathbf{b})$. The reformulation (18) is just a generalisation of the pointwise formulation (17) who has $f_i(\mathbf{b}) = f(x_i, \mathbf{b})$. This has no influence on the least squares analysis to be described presently. In the following we shall assume that the functions f_i are sufficiently smooth in the argument \mathbf{b} .

By introducing the matrix $\mathbf{W} = \text{diag}(w_i)$ and the n -vector $\mathbf{f}(\mathbf{b}) = (f_i(\mathbf{b}))$ we can express (18) in vector notation as follows:

$$\phi(\mathbf{b}) = \|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{f}(\mathbf{b}))\|^2 \quad (19)$$

Here $\|\cdot\|$ denotes the usual Euclidean norm. The corresponding minimization problem reads

$$\phi_{\min} = \min_{\mathbf{b} \in \mathbb{R}^k} \{\|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{f}(\mathbf{b}))\|^2\} \quad (20)$$

A solution \mathbf{b} to (20) satisfies the gradient equation

$$\nabla \phi(\mathbf{b}) = \mathbf{0} \quad (21)$$

which is equivalent to the k equations

$$\frac{\partial \phi(\mathbf{b})}{\partial b_j} = 0, \quad j = 1, \dots, k \quad (22)$$

By (18) and (22) we obtain

$$\sum_{i=1}^n w_i (y_i - f_i(\mathbf{b})) p_{ij} = 0, \quad j = 1, \dots, k \quad (23)$$

where

$$p_{ij} = \frac{\partial f_i(\mathbf{b})}{\partial b_j} \quad (24)$$

It is practical to collect the derivatives (24) in the $n \times k$ matrix

$$\mathbf{P} = (p_{ij}) \quad (25)$$

The equations (23) are called the *normal equations* for the problem. They are in general nonlinear and must be solved iteratively. Solution methods will be discussed in Sections 4.5 and 4.6. Only for linear or linearized models the normal equations are linear.

It is instructive to consider the linear case in some detail. Here (16) takes the form

$$y = \sum_{j=1}^k g_j(x) b_j \quad (26)$$

The x -dependence in $g_j(x)$ is arbitrary and may very well be nonlinear; what matters is that the fitting parameters b_j should enter linearly in the model. The derivatives $p_{ij} = g_j(x_i)$ are independent of b_j , and (18) can be written

$$\phi(\mathbf{b}) = \sum_{i=1}^n w_i \left(y_i - \sum_{j=1}^k p_{ij} b_j \right)^2 \quad (27)$$

The normal equations take the classical form

$$\sum_{j'=1}^k \sum_{i=1}^n w_i p_{ij} p_{ij'} b_{j'} = \sum_{i=1}^n w_i y_i p_{ij}, \quad j = 1, \dots, k \quad (28)$$

The equations (19–20) can be written

$$\phi(\mathbf{b}) = \|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{P}\mathbf{b})\|^2 \quad (29)$$

$$\phi_{\min} = \min_{\mathbf{b} \in \mathbb{R}^k} \{\|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{P}\mathbf{b})\|^2\} \quad (30)$$

The problem (30) is solved by (28) which can be written

$$\mathbf{P}^T \mathbf{W} \mathbf{P} \mathbf{b} = \mathbf{P}^T \mathbf{W} \mathbf{y} \quad (31)$$

where T stands for transpose. For unweighted data we have $\mathbf{W} = \mathbf{I}_n$ (\mathbf{I}_n is the unit matrix of order n), and so

$$\mathbf{P}^T \mathbf{P} \mathbf{b} = \mathbf{P}^T \mathbf{y} \quad (32)$$

Assuming that the coefficient matrix $\mathbf{P}^T \mathbf{W} \mathbf{P}$ in (31) is nonsingular, it must be positive definite too. The same applies to $\mathbf{P}^T \mathbf{P}$ in (32). The case described in (26–32) represents a general *linear regression* model. It is a fundamental building block in NLLS procedures and their statistical analysis.

Returning to the nonlinear case we shall ignore the complications from possible non-uniqueness when solving the normal equations (23). Here we just assume that a usable solution \mathbf{b} can be found.

4.2 Constraints

It is important to be able to impose constraints on the free variation of the model parameters. In principle a constraint could be an equality, $h(\mathbf{b}) = 0$, as well as an inequality $h(\mathbf{b}) \geq 0$, where $h(\mathbf{b})$ is an arbitrary function of the parameter vector.

Although inequality constraints could sometimes be useful, we abandon them in this work because they would lead to quadratic programming problems, and thereby complicate our models considerably. In our algorithm there is, however, a built-in sign check on some of the nonlinear parameters (e.g. annihilation rates). Should an iteration step make such a parameter negative, a new iterate is determined by halving the correction vector from the old one. As a rule, many such “sign excursions” means an inadequate model parameterizing. On the other hand, no sign checks are made on the linear parameters.

Incorporation of general equality constraints would be possible in the framework of our least-squares method. However, apart from trivial single-parameter constraints, $b_j = c$, linear constraints on the linear parameters are sufficient for our purpose, and as we shall see, involve straightforward generalizations of the unconstrained setup discussed previously.

In Section 4.6 we shall describe the separable least-squares technique used in *PALSfit*. The effect of this method is to define subproblems in which the minimization takes place in the space of the linear parameters only. Hence the incorporation of constraints can just as well be discussed in terms of the linear model (26) where $\phi(\mathbf{b})$ is given by (27). In other words, in the constraints analysis we replace k by the number p of linear parameters in the model and consider an all-linear model where \mathbf{b} is replaced by the “linear” parameter vector $\boldsymbol{\alpha} \in \mathbb{R}^p$.

Thus we assume that m independent and consistent linear constraints on the p components of $\boldsymbol{\alpha}$ are given ($m \leq p$):

$$h_{l1}\alpha_1 + \cdots + h_{lp}\alpha_p = \gamma_l, \quad l = 1, \dots, m \quad (33)$$

In vector form (33) reads

$$\mathbf{H}\boldsymbol{\alpha} = \boldsymbol{\gamma} \quad (34)$$

where $\mathbf{H} = (h_{lj})$ is an $m \times p$ matrix and $\boldsymbol{\gamma} = (\gamma_l)$ is an m -vector. Both \mathbf{H} and the augmented matrix $(\mathbf{H}, \boldsymbol{\gamma})$ are of rank m .

A number of technical questions about how the constraints (33) or (34) influence the NLLS procedure will be discussed in Section 4.7.

4.3 Statistical analysis

In this and the following section we address the question of the statistical scatter in the parameters and ϕ_{\min} that can be expected in NLLS parameter estimation.

Suppose the spectrum (y_i) contains experimental values subject to statistical fluctuations, while the weights (w_i) are fixed. Ideally we should imagine an infinite ensemble of similar spectra $\mathbf{y} = (y_i)$ be given. Let us first consider the unconstrained case. Through solution of the normal equations (23) each spectrum \mathbf{y} gives rise to a parameter estimate $\mathbf{b} = \mathbf{b}(\mathbf{y})$. Hence also \mathbf{b} becomes a random (vector) variable with a certain joint distribution.

We shall use the symbol $E[\cdot]$ for expected value (ensemble mean) and $\text{Var}[\cdot]$ for variance. We introduce the “ensemble-mean spectrum”

$$\boldsymbol{\eta} = (\eta_i) = E[\mathbf{y}] \quad (35)$$

and the corresponding hypothetical estimate

$$\mathbf{b}_0 = (b_{j0}) = \mathbf{b}(\boldsymbol{\eta}) \quad (36)$$

Thus \mathbf{b}_0 is the solution of (23) corresponding to the particular spectrum (η_i) . Now, given an arbitrary spectrum (y_i) , let the corresponding parameter vector be $\mathbf{b} = (b_j)$. If we assume that $\mathbf{b} - \mathbf{b}_0 = \Delta\mathbf{b} = (\Delta b_j)$ is so small that our model is locally linear in \mathbf{b} around \mathbf{b}_0 , we have to a first-order approximation

$$f_i(\mathbf{b}) = f_i(\mathbf{b}_0) + \sum_{j=1}^k p_{ij} \Delta b_j \quad (37)$$

where p_{ij} are the derivatives (24) evaluated at \mathbf{b}_0 . We insert (37) into the normal equations (23) and obtain a linear equation system of order k with Δb_j as unknowns. In vector notation this system reads

$$\mathbf{P}^T \mathbf{W} \mathbf{P} \Delta\mathbf{b} = \mathbf{P}^T \mathbf{W} \Delta\mathbf{y} \quad (38)$$

where $\Delta\mathbf{y}$ is a vector with components

$$\Delta y_i = y_i - f_i(\mathbf{b}_0), \quad i = 1, \dots, n \quad (39)$$

We note the similarity with the linear case (31). The system (38) has the solution

$$\Delta\mathbf{b} = \mathbf{K} \Delta\mathbf{y} \quad (40)$$

where

$$\mathbf{K} = (\mathbf{P}^T \mathbf{W} \mathbf{P})^{-1} \mathbf{P}^T \mathbf{W} \quad (41)$$

The *covariance matrix* of a vector variable \mathbf{v} will here be denoted $\boldsymbol{\Sigma}(\mathbf{v})$. (Other names for this matrix are dispersion matrix and variance-covariance matrix, since the diagonal row contains the component variances.) It is well-known that if two vectors \mathbf{v} and \mathbf{w} are related by a linear transformation

$$\mathbf{w} = \mathbf{A} \mathbf{v} \quad (42)$$

then

$$\boldsymbol{\Sigma}(\mathbf{w}) = \mathbf{A} \boldsymbol{\Sigma}(\mathbf{v}) \mathbf{A}^T \quad (43)$$

Our primary goal is to estimate the covariance matrix

$$\boldsymbol{\Sigma}(\mathbf{b}) = (\sigma_{jj'}) \quad (44)$$

Equation (40) shows that $\Delta\mathbf{b}$ is related to $\Delta\mathbf{y}$ by a (locally) linear transformation, and so we obtain from (43) the approximate result

$$\boldsymbol{\Sigma}(\mathbf{b}) = \mathbf{K} \boldsymbol{\Sigma}(\mathbf{y}) \mathbf{K}^T \quad (45)$$

We now assume that the measurements y_i are independent. Let

$$\text{Var}[y_i] = s_i^2, \quad i = 1, \dots, n \quad (46)$$

such that s_i is the standard deviation of y_i . Then $\mathbf{\Sigma}(\mathbf{y}) = \text{diag}(s_i^2)$. We also assume that the variances s_i^2 ($i = 1, \dots, n$) are known, or at least that estimates are available. With this knowledge it is appropriate to use the *statistical weighting* introduced in (2) in Section 2.1. We can show that this leads to a simple form of $\mathbf{\Sigma}(\mathbf{b})$. By using (45) and observing that (2) implies

$$\mathbf{W}\mathbf{\Sigma}(\mathbf{y}) = \mathbf{I}_n \quad (47)$$

we obtain after reduction the formula

$$\mathbf{\Sigma}(\mathbf{b}) = (\mathbf{P}^T \mathbf{W} \mathbf{P})^{-1} \quad (48)$$

which holds at least approximately. It is exact in the linear regression case (31).

Still under the assumption of a locally linear model and of statistical weighting as described, we shall next study the distribution of ϕ_{\min} in (20). Here we make the additional assumptions that we have an *ideal model*, i.e.

$$f_i(\mathbf{b}_0) = \eta_i \quad (49)$$

and that each measurement y_i has a Gaussian distribution,

$$y_i \in N(\eta_i, s_i^2) \quad (50)$$

Then by (39) and (49)

$$\Delta y_i \in N(0, s_i^2) \quad (51)$$

In Section 4.7 it is shown that ϕ_{\min} under these assumptions has a χ^2 -distribution with f degrees of freedom,

$$\phi_{\min} \in \chi^2(f) \quad (52)$$

where f is the number of data values minus the number of fitted parameters,

$$f = n - k \quad (53)$$

For this reason ϕ_{\min} is often called χ^2 . Thus

$$\chi^2 \equiv \phi_{\min} = \min_{\mathbf{b} \in \mathbb{R}^k} \sum_{i=1}^n \left(\frac{y_i - f_i(\mathbf{b})}{s_i} \right)^2 \quad (54)$$

The results derived for $\mathbf{\Sigma}(\mathbf{b})$ and ϕ_{\min} are independent of the applied fitting technique. But we have assumed an unconstrained variation of all components of the k -vector \mathbf{b} . When linear constraints on the linear parameters are included, the analysis still holds for a “basic subset” of k_{free} independent parameter components, as will be shown in Section 4.7. Thus in the distribution (52–53) for ϕ_{\min} we should replace k with k_{free} . To obtain $\mathbf{\Sigma}(\mathbf{b})$ we incorporate the linear constraints (33) or (34) and express the remaining components (deterministically) in terms of the free ones. These operations as well as the resulting formula for $\mathbf{\Sigma}(\mathbf{b})$ are given in Section 4.7.

If the parameter vector \mathbf{b} is transformed to another vector \mathbf{b}_1 before the output is presented, the covariance matrix of \mathbf{b}_1 is computed as

$$\mathbf{\Sigma}(\mathbf{b}_1) = \mathbf{J}\mathbf{\Sigma}(\mathbf{b})\mathbf{J}^T \quad (55)$$

where

$$\mathbf{J} = d\mathbf{b}_1/d\mathbf{b} \quad (56)$$

is the Jacobian of the transformation, cf. (43). In *PALSfit* we use only simple transformations when passing from \mathbf{b} to \mathbf{b}_1 , or no transformation at all. Examples are lifetimes τ_j in ns instead of annihilation rates λ_j in channels⁻¹, and widths in FWHM instead of in standard deviations. These give rise to trivial diagonal elements in \mathbf{J} . On the other hand, the presentation of relative intensities I_j instead of absolute intensities J_j induces a diagonal block in the upper-left corner of \mathbf{J} with the (j, j') -entry $I_j(\delta_{jj'} - I_j)/J_j$.

4.4 Statistical interpretations

In Section 4.3 we discussed the estimation of the covariance matrix

$$\mathbf{\Sigma}(\mathbf{b}) = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1k} \\ \dots & \dots & \dots \\ \sigma_{k1} & \dots & \sigma_{kk} \end{pmatrix} \quad (57)$$

The standard deviations of the estimated parameters are extracted from its diagonal as $\sigma_j = \sqrt{\sigma_{jj}}$, while the off-diagonal entries contain the covariances. In the usual way we construct the correlation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & \dots & \rho_{1k} \\ \dots & \dots & \dots \\ \rho_{k1} & \dots & 1 \end{pmatrix} \quad (58)$$

by the formula

$$\rho_{jj'} = \sigma_{jj'} / (\sigma_j \sigma_{j'}) \quad (59)$$

A consequence of the assumed normal distribution of y_i is that the parameter estimates too will be (approximately) normally distributed and their joint distribution is completely determined by the covariance matrix $(\sigma_{jj'})$. The natural statistical interpretation of $(\sigma_{jj'})$ or $(\sigma_j, \rho_{jj'})$ is an estimate of the covariance structure of the computed parameters in a series of repetitions of the spectrum recording under identical physical conditions. We would expect that $(\sigma_{jj'})$ might show a good deal of scatter in such a series.

The standard deviations and correlation coefficients may be used to compute estimated standard deviations of new parameters that are functions of the primary parameters presented in the output from *PALSfit*, e.g. a mean lifetime or a trapping rate. The standard deviation of such a parameter, z , is given (to a first-order approximation) by:

$$\begin{aligned} \sigma_z &= \left\{ \sum_{j=1}^k \sum_{j'=1}^k \frac{\partial z}{\partial b_j} \frac{\partial z}{\partial b_{j'}} \rho_{jj'} \sigma_j \sigma_{j'} \right\}^{\frac{1}{2}} = \\ &= \left\{ \sum_{j=1}^k \left(\frac{\partial z}{\partial b_j} \right)^2 \sigma_j^2 + 2 \sum_{j=1}^{k-1} \sum_{j'=j+1}^k \frac{\partial z}{\partial b_j} \frac{\partial z}{\partial b_{j'}} \rho_{jj'} \sigma_j \sigma_{j'} \right\}^{\frac{1}{2}} \end{aligned} \quad (60)$$

This result follows by the transformation rule (55).

There is another property of the correlation matrix which might be useful in practice. Suppose an analysis of a given spectrum results in an estimated parameter vector $\mathbf{b} = (b_j)$, $j = 1, \dots, k$. One may ask: What happens to the remaining components if one of them, say b_1 , is forced to be shifted a small amount Δb_1 , and the analysis is repeated with the same spectrum? It can be shown that the other components will be shifted according to the formula

$$\Delta b_j = (\sigma_j / \sigma_1) \rho_{1j} \Delta b_1, \quad j = 2, \dots, k \quad (61)$$

A proof can be found in Section 4.7. The formula (61) refers to a single spectrum and is therefore deterministic. In principle its validity is restricted to small shifts due to the nonlinearity of our models. In our experience the formula is applicable up to at least $\Delta b_1 \approx 2-3 \times \sigma_1$ for well-defined fitting problems with small σ_j . For fits with large σ_j it seems to be valid only up to $\Delta b_1 \approx 0.1-0.2 \times \sigma_1$, and in certain pathological cases it fails completely; such failures may be ascribed to imperfect models or strong nonlinearities.

We saw in Section 4.3 that $\chi^2 = \phi_{\min}$ in (54) under certain assumptions has a χ^2 -distribution with f degrees of freedom, i.e. (52) holds good with

$$f = n - k + m = n - k_{\text{free}} \quad (62)$$

Here m is the number of constraints, so that $k_{\text{free}} = k - m$ is the effective number of free parameters in the estimation. The mean and variance of $\phi_{\min} = \chi^2$ are

$$E[\chi^2] = f \quad (63)$$

and

$$\text{Var}[\chi^2] = 2f \quad (64)$$

From the χ^2 statistics one can derive a “goodness-of-fit” significance test for the validity of the asserted ideal model, cf. (49). In such a χ^2 -test we compute the probability $P\{\chi^2 < \chi_{\text{obs}}^2\}$ that a χ^2 -distributed variable with f degrees of freedom will not exceed the observed value χ_{obs}^2 . A value close to 100% indicates systematic deviation from the assumed model, and we use the phrase “significance of imperfect model” for this probability. We also compute the quantity

$$V = \chi^2/f, \quad (65)$$

with mean

$$\text{E}[V] = 1 \quad (66)$$

and variance

$$\text{Var}[V] = 2/f \quad (67)$$

V is sometimes called the “reduced chi-square” or the “variance of the fit”; with a good fit this quantity should be close to unity.

We conclude this section with some comments on the underlying assumptions in the statistical NLLS analysis which were:

1. Small fluctuations of each data value y_i , i.e. $\text{Var}[y_i]$ small.
2. Our model is only weakly nonlinear in the parameter vector \mathbf{b} .
3. An ideal model which means that (49) holds.
4. The data values y_i are independent.
5. Each y_i has a Gaussian distribution.
6. “Statistical weighting” (2).
7. The population variances $\text{Var}[y_i]$ are known in advance.

Assumptions 1, 2, 3 should be considered together; for example, violation of 1 and 3 may both invalidate the linear approximation (37).

For NLLS problems with strong non-linearities it is well-known that our covariance matrix formula, which is based on linear expansion of the model, may in general produce over-optimistic standard deviation estimates. Nevertheless, such estimates could be useful for qualitative purposes.

Assumption 1 is a fair approximation in PALSfit applications, where it should be understood in the relative meaning; it holds provided the counts y_i are not too small.

Assumption 3 expresses that our model “explains” the observed data perfectly, apart from the inevitable statistical noise. This hypothesis was subject to a chi-square goodness-of-fit test as explained.

Assumption 4 is natural in many applications; however in practice some measurements might show a certain correlation between neighbouring data values.

Assumption 5 is needed only for the analysis of the goodness-of-fit. Many distributions encountered in practice do not deviate much from the normal distribution and thus admits an approximately correct analysis. In particular, this is true for Poisson counting statistics, again provided the counts are large enough.

Regarding Assumption 6, statistical weighting is a convenient means to equalize the impact from the individual observations y_i on the fit. To accomplish it we shall need (estimates of) the variances $\text{Var}[y_i]$ (see also Assumption 7).

Regarding Assumption 7, the theoretical values of the population variances $\text{Var}[y_i]$ are sometimes unavailable and need to be estimated. In some applications the variances are only known up to a constant of proportionality. By using statistical weighting nevertheless, this would not affect the outcome of the NLLS parameter estimation itself. However, the chi-square analysis would not be possible in the usual way due to the lack of normalization.

The way *PALSfit* is affected by Assumptions 6 and 7 is that we do not know $\text{Var}[y_i]$ à priori when setting up the statistical weighting. In the usual case the y_i are unnormalized counts with Poisson statistics, so that $\text{Var}[y_i] \approx y_i$, and (2) may be replaced by

$$w_i = 1/y_i \quad (68)$$

(In extreme cases the discrete nature of y_i may become a problem. This happens when the counts are very small or even zero; in such a case we still make a least-squares fit, by replacing the zero counts by 1, but of course the preconditions for the statistical analysis no longer hold.)

As mentioned in Chapter 2, *PALSfit* uses two least-square iteration cycles such that (68) is used in the first cycle, but in the second cycle y_i is replaced by the preliminary model-predicted value $f_i(\mathbf{b})$, i.e.

$$w_i = 1/f_i(\mathbf{b}) \quad (69)$$

This “weight smoothing” is slightly better than (68), because $f_i(\mathbf{b})$ does not fluctuate as does y_i .

4.5 Marquardt minimization

As mentioned in Section 4.1 the normal equations (23) are in general nonlinear and must be solved iteratively. We now describe such an iterative method called Marquardt’s principle, which is an efficient combination of two classical unconstrained minimization procedures; constraints will be taken care of as described in Sections 4.2 and 4.7.

Basically, we use Newton’s iterative method (other names are the Gauss-Newton or the Taylor series method), which we shall presently explain. However, first we shall prove the following expansion formula which is approximately correct provided \mathbf{d} is small, the fit is good, the model locally linear, and \mathbf{b} is close to the solution of the NLLS problem:

$$\phi(\mathbf{b} + \mathbf{d}) = \phi(\mathbf{b}) + \nabla\phi(\mathbf{b}) \cdot \mathbf{d} + \mathbf{d}^T \mathbf{P}^T \mathbf{W} \mathbf{P} \mathbf{d} \quad (70)$$

with the usual meaning of \mathbf{W} and \mathbf{P} . Using a quadratic Taylor expansion we obtain

$$\phi(\mathbf{b} + \mathbf{d}) = \phi(\mathbf{b}) + \nabla\phi(\mathbf{b}) \cdot \mathbf{d} + \frac{1}{2} \mathbf{d}^T \mathbf{S} \mathbf{d} + \mathcal{O}(\|\mathbf{d}\|^3) \quad (71)$$

Here $\mathbf{S} = \{s_{jj'}\}$ is the Hessian of $\phi(\mathbf{b})$. From the expression (18) we find

$$s_{jj'} = \frac{\partial^2 \phi(\mathbf{b})}{\partial b_j \partial b_{j'}} = 2 \sum_{i=1}^n w_i \left(\frac{\partial f_i}{\partial b_j} \frac{\partial f_i}{\partial b_{j'}} - (y_i - f_i) \frac{\partial^2 f_i}{\partial b_j \partial b_{j'}} \right) \quad (72)$$

with $f_i = f_i(\mathbf{b})$. We shall neglect the term $\sum_{i=1}^n w_i (y_i - f_i) \partial^2 f_i / \partial b_j \partial b_{j'}$ in (72). The reason for doing so is that we expect some cancellation to take place in the summation process, because the residuals $y_i - f_i$ are supposed to fluctuate around zero when the fit is good. We have also assumed that the second derivatives $\partial^2 f_i / \partial b_j \partial b_{j'}$, which express the nonlinearity of the model, are not too large. Hence, approximately

$$\mathbf{S} = 2\mathbf{P}^T \mathbf{W} \mathbf{P} \quad (73)$$

Inserting this in (71) we establish (70). Returning to Newton’s method, let \mathbf{b} be a guessed or previously iterated parameter vector. Newton’s correction step \mathbf{d} now solves the local minimization problem

$$\min_{\mathbf{d} \in \mathbb{R}^k} \{\phi(\mathbf{b} + \mathbf{d})\} \quad (74)$$

where $\phi(\mathbf{b} + \mathbf{d})$ is approximated by (70). For brevity we shall write

$$\mathbf{A} = \mathbf{P}^T \mathbf{W} \mathbf{P} \quad (75)$$

Assuming that \mathbf{P} has full rank, \mathbf{A} will be positive definite. By taking gradients we obtain

$$\nabla \phi(\mathbf{b} + \mathbf{d}) = \nabla \phi(\mathbf{b}) + 2\mathbf{A}\mathbf{d} + \mathcal{O}(\|\mathbf{d}\|^2) \quad (76)$$

We equate this to zero and then compute the Newton step from the normal equation system (cf. (31) and (38))

$$\mathbf{A}\mathbf{d} = \mathbf{g} \quad (77)$$

Here the vector \mathbf{g} is given by

$$\mathbf{g} = -\frac{1}{2}\nabla \phi(\mathbf{b}) \quad (78)$$

According to (18) its components are

$$g_j = \sum_{i=1}^n w_i (y_i - f_i(\mathbf{b})) p_{ij}, \quad j = 1, \dots, k \quad (79)$$

Subsequently $\mathbf{b} + \mathbf{d}$ replaces \mathbf{b} as the new iterate, and the iterations continue. With the pure Newton method we cannot guarantee that the new $\phi = \phi(\mathbf{b} + \mathbf{d})$ is smaller than the old one. Indeed the procedure often tends to diverge due to strong nonlinearities, in particular when the initial guess is bad. To ensure a decrease in ϕ we introduce the Marquardt [32] modification of (77),

$$(\mathbf{A} + \lambda \mathbf{D})\mathbf{d} = \mathbf{g} \quad (80)$$

where \mathbf{D} is a diagonal matrix with the same diagonal row as the positive definite matrix \mathbf{A} . λ is a parameter that is at our disposal. It provides interpolation between Newton's method and a gradient-like method. The former is obtained by setting $\lambda = 0$, cf. (77). On the other hand, when $\lambda \rightarrow \infty$ we obtain a solution vector proportional to $\mathbf{D}^{-1}\mathbf{g}$. According to (78) \mathbf{g} is proportional to the negative gradient vector $-\nabla \phi$, so $\mathbf{D}^{-1}\mathbf{g}$ becomes a scaled version of $-\nabla \phi$ and shares with this the property that ϕ (assumed $> \phi_{\min}$) certainly decreases initially along the correction vector, although it need not have the steepest descent direction. We can now roughly sketch Marquardt's procedure. The equation to be solved at iteration number r reads

$$(\mathbf{A}^{(r)} + \lambda^{(r)} \mathbf{D}^{(r)})\mathbf{d}^{(r)} = \mathbf{g}^{(r)} \quad (81)$$

From its solution $\mathbf{d}^{(r)}$ we calculate

$$\mathbf{b}^{(r+1)} = \mathbf{b}^{(r)} + \mathbf{d}^{(r)} \quad (82)$$

and a new ϕ -value, $\phi^{(r+1)}$. Now it is essential that $\lambda^{(r)}$ is so chosen that

$$\phi^{(r+1)} \leq \phi^{(r)} \quad (83)$$

If we are not already at the minimum, it is always possible to satisfy (83) by selecting a sufficiently large $\lambda^{(r)}$, and so we avoid the divergence problems encountered in Newton's method. However, $\lambda^{(r)}$ should not be chosen unnecessary large, because we then get a small correction vector of gradient-like type which would give slow convergence. In the later iterations, when convergence is approached, λ should be small. Then we approach Newton's method which has a fast (quadratic) rate of convergence near the minimum. The procedure has converged when $\phi^{(r)}$ and $\mathbf{b}^{(r)}$ are stationary with increasing r . For the detailed strategy we refer to Marquardt [32]. The algorithm is sometimes called the Levenberg-Marquardt method (LM) since Levenberg [33] already in 1944 put forward essential parts of the ideas taken up by Marquardt in 1963 [32].

Over the years LM has undergone a number of refinements, adding more robustness to it. In earlier versions of *PALSfit* we used LM as in [32]. But pure LM puts no bounds on the step vector $\mathbf{d}(\lambda)$. Modern LM implementations use a "trust-region" enhancement and replace the unrestricted minimization of ϕ by the quadratic programming problem

$$\min_{\mathbf{d}(\lambda)} \left\{ \phi : \|\mathbf{D}^{1/2} \mathbf{d}(\lambda)\| \leq \Delta \right\} \quad (84)$$

The effect is to restrict the size of $\mathbf{d} = \mathbf{d}(\lambda)$. The bound Δ is adjusted each time a *major* iteration step begins and is decreased if the solution of (84) does not provide a suitable correction. We have adopted this idea for use in *PALSfit* from the work of Moré [34], as implemented in the software package MINPACK-1 [35] for unconstrained least-squares minimization. A subroutine from this package, LMPAR, performs *minor* iterations by finding a value of λ that solves (84) approximately. The optimal λ is saved for use as an initial estimate in the next major step. Details of this technique are found in Moré [34].

4.6 Separable least-squares technique

A substantial gain in computing efficiency can be obtained when some of the k components of the parameter vector \mathbf{b} enter our model linearly. Indeed this is the case in *PALSfit*. The least-squares problem is then called *separable* or *semilinear*. Separable procedures have been studied by several authors [36, 37, 38, 39, 40, 41] and have proved successful in this work and in many other applications as well. In the separable case (16) can be written

$$y = f(x; \mathbf{b}) = f(x; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{j=1}^p \alpha_j f_j(x; \boldsymbol{\beta}) \quad (85)$$

This means a partitioning of the k -vector

$$\mathbf{b} = \begin{matrix} p \\ q \end{matrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \quad (86)$$

into a “linear” p -vector $\boldsymbol{\alpha} = (\alpha_j)$ and a “nonlinear” q -vector $\boldsymbol{\beta} = (\beta_j)$, where $p + q = k$. Corresponding to (85) we have

$$f_i(\mathbf{b}) = \sum_{j=1}^p \alpha_j f_{ij}(\boldsymbol{\beta}) = (\mathbf{F}(\boldsymbol{\beta})\boldsymbol{\alpha})_i, \quad i = 1, \dots, n \quad (87)$$

where $\mathbf{F} = \mathbf{F}(\boldsymbol{\beta})$ is an $n \times p$ matrix with elements $f_{ij} = f_{ij}(\boldsymbol{\beta})$. With these definitions $\mathbf{f}(\mathbf{b})$ entering (20) can be written

$$\mathbf{f}(\mathbf{b}) = \mathbf{F}(\boldsymbol{\beta})\boldsymbol{\alpha} \quad (88)$$

In separable NLLS we consider the linear subproblems of (20) where $\boldsymbol{\beta}$ is fixed and $\boldsymbol{\alpha}$ varies:

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \{ \|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{F}(\boldsymbol{\beta})\boldsymbol{\alpha})\|^2 : \boldsymbol{\beta} \text{ fixed} \} \quad (89)$$

Considering first the unconstrained case, the standard linear least-squares analysis tells that $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\boldsymbol{\beta})$ is the solution of the p th-order normal equation system

$$\mathbf{F}^T \mathbf{W} \mathbf{F} \boldsymbol{\alpha} = \mathbf{F}^T \mathbf{W} \mathbf{y} \quad (90)$$

cf. the linear regression case (26–32). Turning to the determination of the nonlinear part $\boldsymbol{\beta}$ of the parameter vector \mathbf{b} , we realize that an iterative method is needed. In fact, there will be an outer loop, where each step provides a correction vector \mathbf{d} to $\boldsymbol{\beta}$, and an inner procedure which invokes a linear minimization (89–90) each time a new trial value of $\boldsymbol{\beta}$ is chosen. We can formulate the nonlinear outer minimization as follows:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^q} \{ \phi(\boldsymbol{\beta}) \equiv \|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{F}(\boldsymbol{\beta})\boldsymbol{\alpha}(\boldsymbol{\beta}))\|^2 \} \quad (91)$$

We solve (91) by a modified Marquardt procedure as explained in Section 4.5, where \mathbf{b} should be replaced by $\boldsymbol{\beta}$. Indeed, equation (80) takes the form

$$(\mathbf{P}^T \mathbf{W} \mathbf{P} + \lambda \mathbf{D}) \mathbf{d} = \mathbf{P}^T \mathbf{W} (\mathbf{y} - \mathbf{F}(\boldsymbol{\beta})\boldsymbol{\alpha}) \quad (92)$$

where \mathbf{P} is now the $n \times q$ matrix with elements

$$p_{ij} = \frac{\partial f_i}{\partial \beta_j} \quad (93)$$

and \mathbf{D} is a diagonal matrix with the same diagonal row as $\mathbf{P}^T \mathbf{W} \mathbf{P}$.

A crucial point in the separable procedure is the evaluation of (93), which can be accomplished by considering the vector $\mathbf{f} = \mathbf{f}(\mathbf{b})$ in (88). Note that \mathbf{f} depends on $\boldsymbol{\beta}$ directly through \mathbf{F} and indirectly through $\boldsymbol{\alpha}$; hence

$$\frac{\partial \mathbf{f}}{\partial \beta_j} = \frac{\partial \mathbf{F}}{\partial \beta_j} \boldsymbol{\alpha} + \mathbf{F} \frac{\partial \boldsymbol{\alpha}}{\partial \beta_j} \quad (94)$$

To find $\partial \boldsymbol{\alpha} / \partial \beta_j$ we take the derivative of both members of (90). This leads to

$$\mathbf{F}^T \mathbf{W} \mathbf{F} \frac{\partial \boldsymbol{\alpha}}{\partial \beta_j} = \frac{\partial \mathbf{F}^T}{\partial \beta_j} \mathbf{W} (\mathbf{y} - \mathbf{F} \boldsymbol{\alpha}) - \mathbf{F}^T \mathbf{W} \frac{\partial \mathbf{F}}{\partial \beta_j} \boldsymbol{\alpha}, \quad j = 1, \dots, q \quad (95)$$

For an ideal model the term in (95) containing the residual vector $\mathbf{y} - \mathbf{F} \boldsymbol{\alpha}$ is negligible when the minimum is approached, but is important when the current iterate is far from convergence.

Now we can give a summary of the complete strategy for the unconstrained separable minimization of ϕ : Start the outer iterations from a guessed value of $\boldsymbol{\beta}$, and select suitable initial values for λ and the bound Δ . For each outer iteration, solve the linear subproblem (89–90) for $\boldsymbol{\alpha}$ and calculate ϕ . Compute the Jacobian elements $\partial \mathbf{f} / \partial \beta_j$ from (95) and (94), and form \mathbf{P} and \mathbf{D} . Then enter an inner procedure and find near-optimal values of λ and the correction vector to $\boldsymbol{\beta}$, $\mathbf{d} = \mathbf{d}(\lambda)$, using Marquardt's method with Moré's modification. Update the bound Δ , replace $\boldsymbol{\beta}$ by $\boldsymbol{\beta} + \mathbf{d}$, and resume the outer iteration loop. The procedure is finished when ϕ has proved to be stationary.

When implementing our separable algorithm, there is a practical difficulty in handling $\partial \mathbf{f} / \partial \beta_j$ in (94). For each data value we must evaluate a $p \times q$ matrix of scalar derivatives which means altogether $n \times p \times q$ values. To reduce the memory demand we use a packed ("sparse-matrix") scheme for storing only the nonzero derivatives.

Linear constraints on linear model parameters, as they occur in *PALSfit*, are readily integrated in the separable NLLS procedure, cf. Sections 4.2 and 4.7.

The numerical solution of many of the linear-algebraic and optimization subproblems in our algorithm is accomplished by software from the standard packages LINPACK [42] and MINPACK-1 [35]. To accommodate application of this software we found it convenient to rescale the NLLS problem, cf. Section 4.7. In that section we also give some comments on the practical numerical solution of NLLS subproblems.

4.7 Various mathematical and numerical issues

In this section a number of technical details are collected. They all have relevance to the previous sections in this chapter.

Implementation of linear constraints

We consider the constrained linear least-squares problem (cf. (33–34) and (30)),

$$\phi_{\min} = \min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \{ \|\mathbf{W}^{1/2} (\mathbf{y} - \mathbf{P} \boldsymbol{\alpha})\|^2 : \mathbf{H} \boldsymbol{\alpha} = \boldsymbol{\gamma} \} \quad (96)$$

This subproblem is part of the separable NLLS method discussed in Section 4.6, where an optimal linear parameter vector $\boldsymbol{\alpha} \in \mathbb{R}^p$ was computed for a given nonlinear parameter vector $\boldsymbol{\beta} \in \mathbb{R}^q$. Thus the derivative matrix $\mathbf{P} = (p_{ij}) = (\partial f_i / \partial \alpha_j)$ is here of size $n \times p$. One way of handling this constraints problem would be to use Lagrange multipliers. This method was used in early predecessors of *PALSfit*. As a result, the normal equation system (31) was extended to a block matrix system:

$$\begin{pmatrix} \mathbf{P}^T \mathbf{W} \mathbf{P} & \mathbf{H}^T \\ \mathbf{H} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\mu} \end{pmatrix} = \begin{pmatrix} \mathbf{P}^T \mathbf{W} \mathbf{y} \\ \boldsymbol{\gamma} \end{pmatrix} \quad (97)$$

where the vector $2\boldsymbol{\mu}$ contains the Lagrange multipliers. Although (97) is simple enough, there are some drawbacks in this procedure. The coefficient matrix in (97) is not positive definite as was $\mathbf{P}^T \mathbf{W} \mathbf{P}$. This could reduce the numerical stability of the calculations. We also note that the constraints increase the size of the “effective normal equation system” from $p \times p$ to $(p + m) \times (p + m)$. Below we describe an elimination method which is now in use in *PALSfit*. It offers better stability, reduced computer time, and reduced storage demand. Since the rank of \mathbf{H} is m , we can construct a nonsingular matrix by picking m independent columns from \mathbf{H} . A suitable permutation will bring these columns to the m first positions. This can be expressed in terms of a p th-order permutation matrix $\boldsymbol{\Pi}$ by

$$\mathbf{H}\boldsymbol{\Pi} = \mathbf{H}' = \begin{matrix} & m & p-m \\ & \text{---} & \text{---} \\ \text{---} & \mathbf{B} & \mathbf{N} \end{matrix} \quad (98)$$

In the language of linear programming we call \mathbf{B} a “basis matrix” for \mathbf{H} , whereas the columns in \mathbf{N} are called “nonbasic”. Because $\boldsymbol{\Pi}$ is orthogonal, $\boldsymbol{\Pi}\boldsymbol{\Pi}^T = \mathbf{I}_p$, (34) can be written

$$\mathbf{H}'\boldsymbol{\alpha}' = \boldsymbol{\gamma} \quad (99)$$

with

$$\boldsymbol{\alpha}' = \boldsymbol{\Pi}^T \boldsymbol{\alpha} \quad (100)$$

Equation (99) has the complete solution

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha}'_0 + \mathbf{Y}'\mathbf{t} \quad (101)$$

where

$$\boldsymbol{\alpha}'_0 = \begin{matrix} m \\ p-m \end{matrix} \begin{pmatrix} \mathbf{B}^{-1}\boldsymbol{\gamma} \\ \mathbf{0} \end{pmatrix} \quad (102)$$

$$\mathbf{Y}' = \begin{matrix} & p-m \\ m & \text{---} \\ p-m & \text{---} \end{matrix} \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{I}_{p-m} \end{pmatrix} \quad (103)$$

and $\mathbf{t} \in \mathbb{R}^{p-m}$. From (100) we get the complete solution of (34):

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \mathbf{Y}\mathbf{t}, \quad \mathbf{t} \in \mathbb{R}^{p-m} \quad (104)$$

where

$$\boldsymbol{\alpha}_0 = \boldsymbol{\Pi}\boldsymbol{\alpha}'_0 \quad (105)$$

and

$$\mathbf{Y} = \boldsymbol{\Pi}\mathbf{Y}' \quad (106)$$

It is practical to partition $\boldsymbol{\Pi}$ in column sections as follows:

$$\boldsymbol{\Pi} = \begin{matrix} & m & p-m \\ & \text{---} & \text{---} \\ \text{---} & \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2 \end{matrix} \quad (107)$$

Then (105) becomes

$$\boldsymbol{\alpha}_0 = \boldsymbol{\Pi}_1 \mathbf{B}^{-1} \boldsymbol{\gamma} \quad (108)$$

To express (106) we note that

$$\mathbf{N} = \mathbf{H}\boldsymbol{\Pi}_2 \quad (109)$$

and so

$$\mathbf{Y} = (\mathbf{I}_p - \boldsymbol{\Pi}_1 \mathbf{B}^{-1} \mathbf{H}) \boldsymbol{\Pi}_2 \quad (110)$$

We have

$$\mathbf{H}\mathbf{Y} = \mathbf{H}'\mathbf{Y}' = \mathbf{0} \quad (111)$$

and the columns of \mathbf{Y} form a basis of the null space or kernel of \mathbf{H} . Using (104) we can reformulate the constrained p -dimensional problem (96) to an unconstrained $(p-m)$ -dimensional problem:

$$\phi_{\min} = \min_{\mathbf{t} \in \mathbb{R}^{p-m}} \{\|\mathbf{W}^{1/2}(\mathbf{y} - \mathbf{P}\alpha_0 - \mathbf{P}\mathbf{Y}\mathbf{t})\|^2\} \quad (112)$$

We see that this can be derived from (30) by substituting $p-m$ for k , $\mathbf{y} - \mathbf{P}\alpha_0$ for \mathbf{y} , $\mathbf{P}\mathbf{Y}$ for \mathbf{P} , and \mathbf{t} for \mathbf{b} . Thus we can immediately write down the normal equation system for (112) by making the corresponding substitutions in (31):

$$(\mathbf{P}\mathbf{Y})^T \mathbf{W}(\mathbf{P}\mathbf{Y}) \mathbf{t} = (\mathbf{P}\mathbf{Y})^T \mathbf{W}(\mathbf{y} - \mathbf{P}\alpha_0) \quad (113)$$

Next we shall derive an expression for the covariance matrix $\Sigma(\mathbf{b})$ of the total parameter vector $\mathbf{b} = (\alpha, \beta)$ when the constraints (33) or (34) are included. Recalling that $\Sigma(\mathbf{b})$ is independent of the actual fitting method, we can estimate it by perturbing the solution vector \mathbf{b} at the end of iterations. From the normal equation system (113) we deduce in analogy with (37–48) that

$$\Sigma(\mathbf{t}) = \{(\mathbf{P}_\alpha \mathbf{Y})^T \mathbf{W}(\mathbf{P}_\alpha \mathbf{Y})\}^{-1} \quad (114)$$

We have written \mathbf{P}_α for \mathbf{P} since we shall now reserve the notation \mathbf{P} for the $n \times k$ matrix containing derivatives $\partial f_i / \partial b_j$ with respect to *all* the $p+q$ components of $\mathbf{b} = (\alpha, \beta)$. Thus we shall write

$$\mathbf{P} = \begin{matrix} & p & q \\ n & \left(\mathbf{P}_\alpha & \mathbf{P}_\beta \right) \end{matrix} \quad (115)$$

We note that

$$\begin{matrix} & p-m & q \\ n & \left(\mathbf{P}_\alpha \mathbf{Y} & \mathbf{P}_\beta \right) \end{matrix} = \mathbf{P}\mathbf{Z} \quad (116)$$

where \mathbf{Z} is given by

$$\mathbf{Z} = \begin{matrix} & p-m & q \\ p & \left(\begin{matrix} \mathbf{Y} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{matrix} \right) \\ q & \end{matrix} \quad (117)$$

This means that (114) can be extended from \mathbf{t} to (\mathbf{t}, β) as follows:

$$\Sigma(\mathbf{t}, \beta) = \{(\mathbf{P}\mathbf{Z})^T \mathbf{W}(\mathbf{P}\mathbf{Z})\}^{-1} \quad (118)$$

Furthermore, since

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \mathbf{0} \end{pmatrix} + \mathbf{Z} \begin{pmatrix} \mathbf{t} \\ \beta \end{pmatrix} \quad (119)$$

we obtain the result

$$\Sigma(\mathbf{b}) = \mathbf{Z} \{(\mathbf{P}\mathbf{Z})^T \mathbf{W}(\mathbf{P}\mathbf{Z})\}^{-1} \mathbf{Z}^T \quad (120)$$

Distribution of ϕ_{\min}

With $\mathbf{b}_0 = (b_{j0})$ defined in (36) and $\mathbf{b} = (b_j)$ being the solution of the normal equations (23), we obtain the following approximate expression of ϕ_{\min} from the linear expansion (37), which is valid for small $\Delta \mathbf{b} = (b_j - b_{j0})$:

$$\phi_{\min} = \sum_{i=1}^n w_i \left(y_i - \eta_i - \sum_{j=1}^k p_{ij} (b_j - b_{j0}) \right)^2 \quad (121)$$

This can also be written

$$\phi_{\min} = \|\mathbf{W}^{1/2}(\Delta \mathbf{y} - \mathbf{P} \Delta \mathbf{b})\|^2 = (\Delta \mathbf{y} - \mathbf{P} \Delta \mathbf{b})^T \mathbf{W}(\Delta \mathbf{y} - \mathbf{P} \Delta \mathbf{b}) \quad (122)$$

with $\Delta \mathbf{y} = (y_i - \eta_i)$ and \mathbf{P} given by (24–25). By (40–41) ϕ_{\min} becomes a quadratic form in $\Delta \mathbf{y}$:

$$\phi_{\min} = \Delta \mathbf{y}^T \mathbf{B} \Delta \mathbf{y} \quad (123)$$

where \mathbf{B} is found to

$$\mathbf{B} = \mathbf{W} - \mathbf{W}\mathbf{P}(\mathbf{P}^T\mathbf{W}\mathbf{P})^{-1}\mathbf{P}^T\mathbf{W} \quad (124)$$

Defining $u_i = \Delta y_i / s_i$, we see that u_i becomes a standardized normal variable,

$$u_i \in \mathcal{N}(0, 1) \quad (125)$$

Then ϕ_{\min} can be expressed as a quadratic form in $\mathbf{u} = (u_i)$:

$$\phi_{\min} = \mathbf{u}^T \mathbf{C} \mathbf{u} \quad (126)$$

with

$$\mathbf{C} = \mathbf{W}^{-1/2} \mathbf{B} \mathbf{W}^{-1/2} = \mathbf{I}_n - \mathbf{M} \quad (127)$$

where

$$\mathbf{M} = \mathbf{W}^{\frac{1}{2}} \mathbf{P}(\mathbf{P}^T \mathbf{W} \mathbf{P})^{-1} \mathbf{P}^T \mathbf{W}^{\frac{1}{2}} \quad (128)$$

Clearly the matrix \mathbf{M} is of rank k and all its nonzero eigenvalues are unity, as is easily verified by premultiplying $\mathbf{M}\mathbf{x} = \lambda\mathbf{x}$ by $\mathbf{P}^T \mathbf{W}^{1/2}$. Hence there is an orthogonal substitution $\mathbf{u} = \mathbf{Q}\mathbf{z}$ which transforms ϕ_{\min} into a sum of $f = n - k$ squares:

$$\phi_{\min} = \sum_{i=1}^{n-k} z_i^2 \quad (129)$$

where the z_i are independent, and each $z_i \in \mathcal{N}(0, 1)$. This means that ϕ_{\min} has a χ^2 -distribution with f degrees of freedom. If there are m independent linear constraints on the parameters, then the expression (112) demonstrates that the number of degrees of freedom is altered to $f = n - (k - m) = n - k_{\text{free}}$.

Proof of parameter shift formula

We shall here give a proof of the formula (61). In the following we consider $\phi(\mathbf{b})$ with fixed spectrum (y_i). We fix Δb_1 and seek the conditional minimum when the other parameters vary. We shall use the expansion formula (70):

$$\phi(\mathbf{b} + \Delta \mathbf{b}) = \phi(\mathbf{b}) + \nabla \phi(\mathbf{b}) \cdot \Delta \mathbf{b} + \Delta \mathbf{b}^T \mathbf{P}^T \mathbf{W} \mathbf{P} \Delta \mathbf{b} \quad (130)$$

We introduce the vector \mathbf{z} with components $\Delta b_2, \dots, \Delta b_k$. The gradient term in (130) can be written

$$\nabla \phi(\mathbf{b}) \cdot \Delta \mathbf{b} = \frac{\partial \phi}{\partial b_1} \Delta b_1 + \nabla_z \phi(\mathbf{b}) \cdot \mathbf{z} \quad (131)$$

where $\nabla_z \phi(\mathbf{b})$ must be zero. Making the partition

$$\mathbf{P}^T \mathbf{W} \mathbf{P} = \begin{pmatrix} a_{11} & \mathbf{d}^T \\ \mathbf{d} & \mathbf{C} \end{pmatrix} \quad (132)$$

(130) can then be written

$$\phi(\mathbf{b} + \Delta \mathbf{b}) = \phi(\mathbf{b}) + \frac{\partial \phi}{\partial b_1} \Delta b_1 + a_{11} \Delta b_1^2 + 2 \Delta b_1 \mathbf{z}^T \mathbf{d} + \mathbf{z}^T \mathbf{C} \mathbf{z} \quad (133)$$

To minimize (133) we take the derivative with respect to \mathbf{z} . After equating the result to zero we deduce that

$$\mathbf{z} = -\Delta b_1 \mathbf{C}^{-1} \mathbf{d} \quad (134)$$

Next we make the similar partitioning

$$\Sigma(\mathbf{b}) = \begin{pmatrix} \sigma_{11} & \mathbf{s}^T \\ \mathbf{s} & \mathbf{\Gamma} \end{pmatrix} \quad (135)$$

We assume statistical weighting which implies the identity $\mathbf{P}^T \mathbf{W} \mathbf{P} \Sigma(\mathbf{b}) = \mathbf{I}_k$, cf. (48). From this we infer that

$$\mathbf{C}^{-1} \mathbf{d} = -\frac{1}{\sigma_{11}} \mathbf{s} \quad (136)$$

Inserting this in (134) yields

$$\mathbf{z} = \frac{\Delta b_1}{\sigma_{11}} \mathbf{s} \quad (137)$$

which proves formula (61) since \mathbf{s} has components $\sigma_{21}, \dots, \sigma_{k1}$.

Scaling in separable NLLS

In order to facilitate the application of standard minimization software we found it appropriate to make a scaling of the problem formulation. We recast the original minimization problem (20) to

$$\phi_{\min} = \min_{\mathbf{b} \in \mathbb{R}^k} \{\|\mathbf{r}(\mathbf{b})\|^2\} \quad (138)$$

where $\mathbf{r}(\mathbf{b})$ is a (scaled) *residual vector* with components

$$r_i = w_i^{1/2}(y_i - f_i(\mathbf{b})) \quad (139)$$

This induces a number of vector and matrix transformations containing the matrix scaling factor $\mathbf{W}^{1/2}$:

$$\mathbf{z} = \mathbf{W}^{1/2} \mathbf{y} \quad (140)$$

$$\mathbf{e} = \mathbf{W}^{1/2} \mathbf{f} \quad (141)$$

$$\mathbf{E} = \mathbf{W}^{1/2} \mathbf{F} \quad (142)$$

$$\mathbf{G} = \mathbf{W}^{1/2} \mathbf{P} \quad (143)$$

Then the counterparts of (88–92) become:

$$\mathbf{e}(\mathbf{b}) = \mathbf{E}(\boldsymbol{\beta}) \boldsymbol{\alpha} \quad (144)$$

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^p} \{\|\mathbf{z} - \mathbf{E}(\boldsymbol{\beta}) \boldsymbol{\alpha}\|^2 : \boldsymbol{\beta} \text{ fixed}\} \quad (145)$$

$$\mathbf{E}^T \mathbf{E} \boldsymbol{\alpha} = \mathbf{E}^T \mathbf{z} \quad (146)$$

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^q} \{\phi(\boldsymbol{\beta}) \equiv \|\mathbf{z} - \mathbf{E}(\boldsymbol{\beta}) \boldsymbol{\alpha}(\boldsymbol{\beta})\|^2\} \quad (147)$$

$$(\mathbf{G}^T \mathbf{G} + \lambda \mathbf{D}) \mathbf{d} = \mathbf{G}^T (\mathbf{z} - \mathbf{E} \boldsymbol{\alpha}) \quad (148)$$

Moreover (94–95) are replaced by:

$$\frac{\partial \mathbf{e}}{\partial \beta_j} = \frac{\partial \mathbf{E}}{\partial \beta_j} \boldsymbol{\alpha} + \mathbf{E} \frac{\partial \boldsymbol{\alpha}}{\partial \beta_j} \quad (149)$$

$$\mathbf{E}^T \mathbf{E} \frac{\partial \boldsymbol{\alpha}}{\partial \beta_j} = \frac{\partial \mathbf{E}^T}{\partial \beta_j} (\mathbf{z} - \mathbf{E} \boldsymbol{\alpha}) - \mathbf{E}^T \frac{\partial \mathbf{E}}{\partial \beta_j} \boldsymbol{\alpha} \quad (150)$$

We see that the effect of these transformations is to “hide” the weights w_i entirely. To include the effect of the linear constraints on (149–150) we use (104) and find:

$$\frac{\partial \mathbf{e}}{\partial \beta_j} = \frac{\partial \mathbf{E}}{\partial \beta_j} \boldsymbol{\alpha} + \mathbf{E} \mathbf{Y} \frac{\partial \mathbf{t}}{\partial \beta_j} \quad (151)$$

$$(\mathbf{E} \mathbf{Y})^T \mathbf{E} \mathbf{Y} \frac{\partial \mathbf{t}}{\partial \beta_j} = \frac{\partial (\mathbf{E} \mathbf{Y})^T}{\partial \beta_j} (\mathbf{z} - \mathbf{E} \boldsymbol{\alpha}) - (\mathbf{E} \mathbf{Y})^T \frac{\partial \mathbf{E}}{\partial \beta_j} \boldsymbol{\alpha} \quad (152)$$

QR decomposition

A direct solution of normal equations, even by Choleski decomposition, may present numerical difficulties inherent with the ill-conditioning of the positive-definite coefficient matrix, say $\mathbf{E}^T \mathbf{E}$ in (146). Instead we use a procedure based on the so-called QR decomposition of \mathbf{E} , viz.

$$\mathbf{E} = \mathbf{Q} \mathbf{R} \quad (153)$$

where \mathbf{Q} is an $n \times p$ matrix with orthonormal columns and \mathbf{R} is a $p \times p$ upper triangular matrix (see, e.g., Chapter 9 in [42]). Using (153) the system (146) is reformulated to $\mathbf{R} \boldsymbol{\alpha} = \mathbf{Q}^T \mathbf{z}$, which can be easily solved by back-substitution. The same procedure is used when solving (150) for $\partial \boldsymbol{\alpha} / \partial \beta_j$, with \mathbf{R} being saved after the solution of (146).

The numerical computation of the covariance matrix $\Sigma(\mathbf{b})$ can also be done by QR technique. We start with the expression (120) which includes the constraints and performs the usual scaling by defining $\mathbf{T} = \mathbf{W}^{1/2}\mathbf{P}$. Then we obtain

$$\Sigma(\mathbf{b}) = \mathbf{Z}((\mathbf{T}\mathbf{Z})^T\mathbf{T}\mathbf{Z})^{-1}\mathbf{Z}^T \quad (154)$$

and can now make the decomposition (of course the factors are different from those in (153)),

$$\mathbf{T}\mathbf{Z} = \mathbf{Q}\mathbf{R} \quad (155)$$

which leads to

$$\Sigma(\mathbf{b}) = \mathbf{Z}\mathbf{R}^{-1}(\mathbf{Z}\mathbf{R}^{-1})^T \quad (156)$$

In some ill-conditioned problems the diagonal row of \mathbf{R} may contain very small elements, which would render the evaluation of $\Sigma(\mathbf{b})$ by (156) completely erratic. There exists a variant of the QR decomposition with column scaling and pivoting that admits a judicious discarding rule for insignificant elements in the \mathbf{R} -diagonal [42, 35]. Following this idea, we shall replace (155) with

$$\mathbf{T}\mathbf{Z}\mathbf{\Lambda}\mathbf{\Pi} = \mathbf{Q}\mathbf{R} \quad (157)$$

where $\mathbf{\Lambda}$ is a diagonal scaling matrix, $\mathbf{\Pi}$ a permutation matrix, and the diagonal elements of \mathbf{R} are in non-increasing order of magnitude. The entries in $\mathbf{\Lambda}$ are chosen as the inverse Euclidean norms of the column vectors of $\mathbf{T}\mathbf{Z}$ and might be called “uncoupled standard deviations”. Instead of (156) we obtain

$$\Sigma(\mathbf{b}) = \mathbf{Z}\mathbf{\Lambda}\mathbf{\Pi}\mathbf{R}^{-1}(\mathbf{Z}\mathbf{\Lambda}\mathbf{\Pi}\mathbf{R}^{-1})^T \quad (158)$$

The expression (158) is only used for the “significant” parameters which corresponds to the upper part of \mathbf{R} . The variance of the “insignificant” parameters are estimated by their uncoupled standard deviations, while the covariance calculation for such parameters are abandoned. (It is easier to grasp the essential features of this procedure if we simplify and replace $\mathbf{T}\mathbf{Z}$ with $\mathbf{P} = \mathbf{Q}\mathbf{R}$.)

5 Mathematical model details

In Sections 2.4 and 2.6 we gave a short presentation of the theoretical models used in *PositronFit* and *ResolutionFit*. Below we shall try to fill the gap between the rather brief description given there of the underlying mathematical models, and the least-squares theory in Chapter 4.

5.1 *PositronFit*

Writing formula (3) as

$$f(t) = \sum_{j=1}^{k_0} \sum_{p=1}^{k_g} \omega_p(a_j * G_p)(t) + B \quad (159)$$

we must evaluate the convolution integral

$$(a_j * G_p)(t) = \int_{-\infty}^{\infty} a_j(v)G_p(t-v)dv \quad (160)$$

where a_j and G_p were defined in (4) and (6), respectively. Henceforward, we prefer to describe the decay of a lifetime component in terms of the annihilation rate

$$\lambda_j = 1/\tau_j \quad (161)$$

instead of the lifetime τ_j itself. It can be shown that

$$(a_j * G_p)(t) = \frac{1}{2} A_j \varphi(t - T_0 - \Delta t_p, \lambda_j, \sigma_p) \quad (162)$$

The function φ acts as a building block for our lifetime spectral model and is defined by

$$\varphi(u, \lambda, \sigma) = \exp\left(-\lambda u + \frac{1}{2} \lambda^2 \sigma^2\right) \operatorname{erfc}\left(\frac{\lambda \sigma^2 - u}{\sqrt{2} \sigma}\right) \quad (163)$$

where erfc stands for the complementary error function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) \quad (164)$$

and erf in turn is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt \quad (165)$$

Inserting (162) in (159) we get

$$f(t) = \frac{1}{2} \sum_{j=1}^{k_0} A_j \sum_{p=1}^{k_g} \omega_p \varphi(t - T_0 - \Delta t_p, \lambda_j, \sigma_p) + B \quad (166)$$

Finally, we compute the integrated model-predicted count f_i over channel i defined by equation (9) in Section 2.4. We use the identity

$$\int \varphi(u, \lambda, \sigma) du = -\frac{1}{\lambda} \left(\varphi(u, \lambda, \sigma) + \operatorname{erfc}\left(\frac{u}{\sqrt{2} \sigma}\right) \right) \quad (167)$$

and obtain

$$f_i = \sum_{j=1}^{k_0} F_{ij} + B = \sum_{j=1}^{k_0} \alpha_j f_{ij} + B \quad (168)$$

where

$$\alpha_j = A_j / \lambda_j = A_j \tau_j \quad (169)$$

is the area or “absolute intensity” ascribed to lifetime component j ,

$$\begin{aligned} f_{ij} = & \frac{1}{2} \sum_{p=1}^{k_g} \omega_p \{ \varphi(t_{ip}, \lambda_j, \sigma_p) - \varphi(t_{i+1,p}, \lambda_j, \sigma_p) \\ & + \operatorname{erfc}(t_{ip}/(\sqrt{2} \sigma_p)) - \operatorname{erfc}(t_{i+1,p}/(\sqrt{2} \sigma_p)) \} \end{aligned} \quad (170)$$

and where we use the shorthand notation

$$t_{ip} = t_i - T_0 - \Delta t_p \quad (171)$$

By now we have arrived at the model expression $f_i = f_i(\mathbf{b})$ entering the least-squares formulation of the fitting problem given in Chapter 4. We also see that (168) is separable as required; the parameter vector \mathbf{b} splits into a “linear” parameter $\boldsymbol{\alpha}$ and a “nonlinear” one $\boldsymbol{\beta}$ given by

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k_0}, B) \quad (172)$$

and

$$\boldsymbol{\beta} = (\lambda_1, \dots, \lambda_{k_0}, T_0) \quad (173)$$

Thus the separable fitting theory of Section 4.6 applies. To perform the computations outlined there, we must evaluate the derivatives of f_{ij} in (170) with respect to λ_j and T_0 ; this job is facilitated by the following two formulas:

$$\frac{\partial \varphi}{\partial u} = -\lambda \varphi(u, \lambda, \sigma) + \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} \exp\left(-\frac{u^2}{2\sigma^2}\right) \quad (174)$$

and

$$\frac{\partial \varphi}{\partial \lambda} = (\lambda \sigma^2 - u) \varphi(u, \lambda, \sigma) - \sqrt{\frac{2}{\pi}} \sigma \exp\left(-\frac{u^2}{2\sigma^2}\right) \quad (175)$$

In Section 2.4 we mentioned the types of constraints which could be imposed on the parameters in *PositronFit*. Those constraints that fix one of the “primary” fitting parameters listed in (172) and (173) are realized by deleting the corresponding components from α or β . This may apply to B , λ_j , and T_0 . Constraints of the type “fixed relative intensity” are not of this simple type because the relative intensities $\alpha_j / \sum \alpha_{j'}$ are not primary parameters. But obviously such constraints are expressible as linear constraints on the linear parameters α_j , i.e. relations of the form

$$\sum h_{ij} \alpha_j = \gamma_i \quad (176)$$

where h_{ij} are known coefficients, cf. (33). The same holds good for constraints of the type “a linear combination of the relative intensities = 0”, as well as the total-area constraint (13).

Regarding the practical computation of (163), that formula may from a numerical point of view be dangerous to use as it stands. The difficulty arises when $u \ll 0$. Then $\varphi(u, \lambda, \sigma)$ itself is small; nevertheless, the first factor of (163) is large and may cause an overflow on the computer. At the same time, the second factor is very small and likely to underflow. A better alternative is to restate (163) as

$$\varphi(u, \lambda, \sigma) = \exp\left(-\frac{u^2}{2\sigma^2}\right) \text{erfc}\left(\frac{\lambda\sigma^2 - u}{\sqrt{2}\sigma}\right) \quad (177)$$

where erfc stands for the normalized complementary error function

$$\text{erfc}(x) = \exp(x^2) \text{erf}(x) \quad (178)$$

It is not hard to develop robust and accurate numerical approximations for this slowly varying function, which is decreasing when $x > 0$ and behaves asymptotically as $1/(\sqrt{\pi}x)$.

5.2 ResolutionFit

Although the basic model in *ResolutionFit* is the same as in *PositronFit*, there are certain differences regarding which parameters enter as fitting parameters since the widths σ_p and the shifts Δt_p are fitting parameters in *ResolutionFit*. Hence (173) should be replaced by

$$\beta = (\lambda_1, \dots, \lambda_{k_0}, T_0, \sigma_1, \dots, \sigma_{k_g}, \Delta t_1, \dots, \Delta t_{k_g}) \quad (179)$$

In Section 2.6 we mentioned the types of constraints which could be imposed on the parameters in *ResolutionFit*. Some of the parameters of (179) can be fixed and in that case should be deleted from β . This may apply to λ_j , σ_p , and Δt_p . Notice that T_0 in *ResolutionFit* is always a free parameter. As a consequence, we must require that at least one of the shifts Δt_p be fixed.

In addition to (174) and (175), we shall also need the derivative of φ with respect to σ :

$$\frac{\partial \varphi}{\partial \sigma} = \lambda^2 \sigma \varphi(u, \lambda, \sigma) - \sqrt{\frac{2}{\pi}} \left(\lambda + \frac{u}{\sigma^2} \right) \exp\left(-\frac{u^2}{2\sigma^2}\right) \quad (180)$$

In *ResolutionFit* we compute shape parameters for the fitted resolution curve. This leads to nonlinear equations involving the φ function of (163). We use a Newton–Raphson procedure for the numerical solution of these.

6 Control files

When running *PALSfit* the program produces a control file (see Fig. 1 in Chapter 1), which contains all the input instructions for either a *PositronFit* or a *ResolutionFit* analysis.

Normally the editing of this file is a task which is done automatically by using the *PALSfit* menus. Nevertheless there might be situations where an inspection or an external editing of the file is appropriate.

In Chapter 1 it was mentioned that in certain situations (batch processing or running under Linux) it may be useful to run the command-driven PATFIT programs *PositronFit* and *ResolutionFit* directly. In that case you will also need to know the structure of the input files [2]. Note that PATFIT and *PALSfit* are input-compatible.

Anyway, the knowledge of the structure of the control files may give the user a good overview of the capabilities of *PALSfit*. Therefore, in the following we shall describe the contents of the control files for *PositronFit* and *ResolutionFit* in some detail. For convenience, we will in some cases use parameter names that occur in the programs.

The control file is composed of one or more control data sets. A control data set is partitioned into a number of data blocks, corresponding roughly to the menus in *PALSfit*. Each block is initiated by a so-called block header. For example, the first block header reads

```
POSITRONFIT DATA BLOCK: OUTPUT OPTIONS
```

in the case of *PositronFit*, and similarly for *ResolutionFit*.

6.1 *PositronFit* control file

A sample *PALSfit* control file for *PositronFit* with a single control data set is shown below.

```
POSITRONFIT DATA BLOCK: OUTPUT OPTIONS
0000
POSITRONFIT DATA BLOCK: SPECTRUM
    512
((10f7.0))
Mol_Crystal spectra.dat
32800 CYCLOOCTANE
    1 0
32800 CYCLOOCTANE
32800   764   778   728   754   755   733   725   743   769
    700   735   694   737   714   766   756   692   741   728
    705   715   729   716   717   723   699   717   719   702
    696   729   637   710   678   745   702   708   686   708
    708   670   677   712   679   664   642   708   718   720
    677   688   722   704   705   701   651   675   703   720
    696   710   723   683   628   746   725   647   730   703
    677   688   691   700   730   676   658   708   714   690
    719   705   707   708   673   701   684   689   692   697
    743   688   721   646   712   716   665   707   730   690
    706   743   713   682   758   720   683   712   679   700
    692   699   692   708   742   740   742   690   708   727
    770   704   766   683   743   822   1049   1810   4146   11412
30056  71498 151210 277754 433291 581716 677357 705264 669401 595116
509460 426352 354314 294562 245828 208141 175625 151785 131162 114887
100962  90149  80698  73201  66518  60971  56076  52147  48976  45968
 43147  40678  39125  36522  35059  33784  32103  30988  30125  29086
 27905  26888  25935  25090  24441  23366  22792  22072  21308  21178
 20144  19616  18959  18302  17973  17247  16784  16432  15678  15607
 15241  14667  14085  13685  13468  12871  12522  12277  11915  11533
 11407  11174  10807  10156  9894  9835  9590  9237  9142  8789
  8694  8424  8249  7828  7537  7623  7368  7025  6916  6836
  6748  6475  6307  6079  6018  5730  5628  5619  5377  5274
  5190  5019  4829  4747  4560  4529  4541  4336  4194  4083
  4007  3865  3855  3751  3701  3568  3546  3390  3363  3262
  3218  3061  3007  2932  2786  2918  2759  2738  2722  2611
  2631  2524  2551  2533  2363  2437  2350  2220  2214  2157
  2026  2155  2008  2009  1998  2018  1898  1847  1834  1764
  1774  1724  1747  1639  1728  1572  1601  1581  1679  1566
  1512  1501  1486  1477  1480  1359  1404  1406  1335  1397
```

```

1337 1299 1258 1240 1281 1240 1203 1195 1217 1159
1117 1163 1079 1049 1077 1080 1038 1086 1020 1051
974 968 1039 996 1001 1018 967 957 964 927
975 953 885 942 921 915 911 935 859 851
880 873 945 863 844 827 852 806 822 838
830 802 825 825 810 774 772 784 796 801
803 799 815 760 796 778 768 752 733 760
719 744 765 786 729 717 719 771 688 730
744 742 672 733 756 719 689 749 686 754
686 693 716 830 716 705 666 743 753 734
710 715 737 705 670 712 650 758 705 764
778 746 688 728 662 716 707 697 733 660
687 768 704 690 644 730 692 691 694 675
697 695 678 675 710 685 767 711 642 683
723 702 732 703 678 686 694 693 745 671
701 699 676 671 659 691 672 669 677 682
701 752 664 651 620 670 717 708 616 700
655 706 791 652 665 670 726 722 713 682
662 650 651 712 680 694 721 676 677 671
694 701 660 696 711 715 682 635 705 715
674 676 680 681 734 737 721 699 675 717
694 667
POSITRONFIT DATA BLOCK: CHANNEL RANGES. TIME SCALE. TIME-ZERO.
35
512
136
501
0.077300
G
137.300
POSITRONFIT DATA BLOCK: RESOLUTION FUNCTION
3
0.5395 0.3539 0.4036
20.000 20.000 60.000
0.0412 -0.0749 0.0000
POSITRONFIT DATA BLOCK: LIFETIMES AND INTENSITY CONSTRAINTS
3
FGG
0.2300 0.4000 2.6500
1
3
28.0000
POSITRONFIT DATA BLOCK: BACKGROUND CONSTRAINTS
2
680.0000
POSITRONFIT DATA BLOCK: AREA CONSTRAINTS
1
35
512
POSITRONFIT DATA BLOCK: SOURCE CORRECTION
2
0.3840 0.9056
8.0000 0.4527
100.0000
1
4
GGGG
0.1200 0.3600 1.2000 2.8000
-1
-3.0000 0.0000 1.0000 1.0000

```

Block 1 contains output options. Apart from the block header there is only one record. It contains four integer keys. Each key is either 0 or 1. The value 1 causes some output action to be taken, whereas 0 omits the action. The actions of the 4 keys are:

1. Write input echo to result file
2. Write each iteration output to result file
3. Write residual plot to result file

4. Write correlation matrix to result file

Regardless of the setting of these keys, *PositronFit* always produces the Main Output.

Block 2 contains the spectrum. The first record (after the block header) contains the integer NCH, which is the total number of channels in the spectrum. Next record contains a description of precisely how the spectrum values are “formatted” in the file—expressed as a so-called FORMAT in the programming language FORTRAN [43]; an asterisk * means free format. After this, two text records follow. In the first a name of a spectrum file is given. (Even when INSPEC = 1 (see below) this name should be present, but is in that case not used by the program.) In the other record an identification label of the spectrum is given. The next record contains two integers, INSPEC and XXX, both taking a value of either 0 or 1. (XXX is specific for the PALSfit/GUI and is not used by PATFIT.) INSPEC = 1 means that the spectrum is an intrinsic part of the present control file. In this case the next record should be a text line with a description of the spectrum. The subsequent records are supposed to hold the NCH spectrum values. On the other hand, INSPEC = 0 means that the spectrum is expected to reside in an external file with the spectrum file name entered above. The program tries to open this file and scans it for a record whose start matches the identification label. After a successful match, the matching (text) line and the spectrum itself is read from the subsequent records in exactly the same way as in the case INSPEC = 1. If XXX = 1 all spectra in the spectrum file will be analysed with the parameters defined in the present data set. If XXX = 0, only the spectrum defined in the present data set will be analysed.

Block 3 contains information related to the measuring system. The first two records (after the block header) contain two channel numbers ICHA1 and ICHA2. These numbers are lower and upper bounds for the definition of a total area range. The next two records contain also two channel numbers ICHMIN and ICHMAX. These define in the same way the channel range which is used in the least-squares analysis. The next record contains the channel width (in ns). The last two records in this block deal with T_0 (time=0 channel number). First comes a constraint flag being either a G or an F. G stands for guessed (i.e. free) T_0 , F stands for fixed T_0 . The other record contains the initial (guessed or fixed) value of T_0 .

Block 4 contains input for definition of the resolution function. The first record (after the block header) contains the number k_g of Gaussian components in the resolution function. Each of the next three records contains k_g numbers. In the first we have the full widths at half maxima of the Gaussians (in ns), FWHM $_j$, $j = 1, \dots, k_g$, in the second their relative intensities (in percent) ω_j , $j = 1, \dots, k_g$, and in the third their peak displacements (in ns) Δt_j , $j = 1, \dots, k_g$.

Block 5 contains data for the lifetime components in the lifetime spectrum as well as constraints on their relative intensities. The first record (after the block header) holds the number k_0 of lifetime components assumed in the model. Each of the next two records contains k_0 data. In the first we have the constraint flags (G = guessed, F = fixed) for the lifetimes. The other record contains the initial values (guessed or fixed) of the k_0 lifetimes. After this comes a record with an integer m telling the number and type of intensity constraints. $|m|$ is equal to the number of constraints, but m itself may be positive, negative, or zero. If $m = 0$ there is no further input data in this block. If $m > 0$, m of the relative intensities are fixed. In this case the next data item is a pair of records with the numbers j_l , $l = 1, \dots, m$ and I_{j_l} , $l = 1, \dots, m$; here j_l is the term number (the succession agreeing with the lifetimes on the previous record) associated with constraint number l , and I_{j_l} is the corresponding fixed relative intensity (in percent). If $m < 0$, $|m|$ linear combinations of the intensities are equal to zero. In this case $|m|$ records follow, each containing the k_0 coefficients h_{lj} , $j = 1, \dots, k_0$ to the intensities for one of the linear combinations, cf. equation (12) in Section 2.4.

Block 6 contains data related to the background. The first record (after the block header) contains an integer indicator KB, assuming one of the values 0, 1, or 2. KB = 0 means a free background; in this case no more data follows in this block. If KB = 1 the background is fixed to the spectrum average from channel ICHBG1 to channel ICHBG2. These two channel numbers follow on the next two records. If KB = 2, the background is fixed to an input value which is entered on the next record.

Block 7 contains input for constraining the total area. The first record (after the block header) holds an integer indicator KAR, assuming one of the values 0, 1, or 2. KAR = 0 means no area constraint; in this case no more data follows in this block. If KAR > 0, the area between two specified channel limits ICHBEG and ICHEND will be fixed, and these channel numbers follow on the next two records. If KAR = 1, the area is fixed to the measured spectrum, and no more input will be needed. If KAR = 2 the area is fixed to an input value which is entered on the next record.

Block 8 contains source correction data. The first record (after the block header) contains an integer k_s denoting the number of components in the source correction spectrum. $k_s = 0$ means no source correction, in which case the present block contains no more data. The next record contains the lifetimes τ_j^s , $j = 1, \dots, k_s$, and the following the relative intensities I_j^s , $j = 1, \dots, k_s$ for the source correction terms. On the next record is the number α which is the percentage of positrons that annihilate in the source, cf. equation (15) in Section 2.4. Then there follows a record with an integer ISEC. When ISEC = 0 the new iteration cycle after the source correction starts from lifetime guesses equal to the converged values from the first (correction-free) cycle. ISEC = 1 tells that the second cycle starts from new input data. These 2nd-cycle input data are now entered in exactly the same way as the 1st-cycle data in Block 5. ISEC = 2 works as ISEC = 1, but with the additional possibility of changing the status of T_0 ; in this case two more records follow, the first containing the constraint flag (G = guessed, F = fixed) for T_0 and the second the value of T_0 .

With the end of Block 8 the entire *PositronFit* control data set is completed. However, as previously mentioned, *PALSfit* accepts multiple control data sets in the same *PositronFit* control file.

6.2 *ResolutionFit* control file

A sample *PALSfit* control file for *ResolutionFit* with a single control data set is shown below.

```

RESOLUTIONFIT DATA BLOCK: OUTPUT OPTIONS
0000
RESOLUTIONFIT DATA BLOCK: SPECTRUM
1023
(/,(10f7.0))
.\Metal defects spectra.DAT
39699 CU-ANNEALED
0 0
RESOLUTIONFIT DATA BLOCK: CHANNEL RANGES. TIME SCALE. TIME-ZERO.
5
1000
140
500
0.013400
173.000
RESOLUTIONFIT DATA BLOCK: RESOLUTION FUNCTION
3
GGG
0.2600    0.3000    0.4000
77.000    19.000    4.000
FGG
0.0000    0.0223   -0.0462

```



```

RESOLUTIONFIT DATA BLOCK: LIFETIMES AND INTENSITY CONSTRAINTS
3
FFG
0.1100    0.1800    0.4000
0
RESOLUTIONFIT DATA BLOCK: BACKGROUND CONSTRAINTS
1
650
1000

```

Block 1 contains output options. It is identical to the corresponding block in the *PositronFit* control file (but of course the name RESOLUTIONFIT must appear in the block header).

Block 2 contains the spectrum. It is identical to the corresponding block in the *PositronFit* control file.

Block 3 contains information related to the measuring system. The first two records (after the block header) contain two channel numbers ICHA1 and ICHA2. These numbers are lower and upper bounds for the definition of a total area range. The next two records contain also two channel numbers ICHMIN and ICHMAX. These define in the same way the channel range which is used in the least-squares analysis. The next record contains the channel width (in ns). The last record in this block contains the initial (guessed) value of T_0 .

Block 4 contains input for definition and initialization of the resolution function. The first record (after the block header) contains the number k_g of Gaussian components in the resolution function. Each of the next two records contains k_g data. In the first we have the constraint flags (G=guessed, F=fixed) for the Gaussian widths. The second contains the initial values (guessed or fixed) of the full widths at half maxima of the Gaussians (in ns), $\text{FWHM}_j^{\text{ini}}$, $j = 1, \dots, k_g$. The next record contains the k_g Gaussian component intensities in percent, ω_j , $j = 1, \dots, k_g$. The last two records in the block contain again k_g data each. First, we have the constraint flags (G=guessed, F=fixed) for the Gaussian shifts; notice that not all the shifts can be free. Next, we have the initial (guessed or fixed) peak displacements (in ns), Δ_j^{ini} , $j = 1, \dots, k_g$.

Block 5 contains data for the lifetime components in the lifetime spectrum as well as constraints on their relative intensities. It is identical to the corresponding block in the *PositronFit* control file.

Block 6 contains data related to the background. It is identical to the corresponding block in the *PositronFit* control file.

This completes the *ResolutionFit* control data set. Multiple data control data sets can be handled in the same way as for *PositronFit*.

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